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PROBLEMS WITH RATIONAL EXPONENTS IN ELEMENTARY MATHEMATICS

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Abstract. The concept and definitions of the common power notation used in elementary mathematics will be discussed with a focus on formalizing an introductory definition and avoiding misconceptions by students and teachers alike.

1. Introduction

The common power notation b^r, with rational exponent r, works fine as long as b is positive and we are considering only the positive value of the power. But as soon as we allow the base b to be negative or imaginary, or even with a positive base b, if we are to consider more than just positive values of its powers, then there are many pitfalls, where the rules or even the familiar definitions fail to work. For example, one common error made by students in dealing with powers and roots is of the following type: $\sqrt{(-8)^2} = -8$ [1, Misconception 9, pp 5–6]. This result obtained by students may be based on the following sequence of calculations:

$$\sqrt{(-8)^2} = (-8)^{\frac{2}{2}} = (-8)^1 = -8,$$

or more simply put, by a cancellation of power and root:

$$\sqrt{(-8)^{2}} = -8.$$

Each step of the calculation above seems valid in view of the definitions of root and power, yet the result is invalid. We will try to analyze the root cause of the misconceptions that underlie errors of this type. For that purpose we reexamine definitions of fractional powers and problem situations associated with them, en route to consideration of rational exponents. Then we will suggest that if some extra care is paid by the textbook authors and instructors when treating powers and roots, such misconceptions can be largely avoided.

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2. Summary of Known Results

Before starting, we fix some notation and state a few results that we use in our discussion. Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural, integer, rational, real and complex numbers, respectively. Also, if K is a ring / field, we use K^* to denote the set of all elements of K that have multiplicative inverses in K. Thus, if K is a field, for example $K = \mathbb{C}$ or $K = \mathbb{R}$, then K^* consists of all non-zero elements of K.

Proposition 2.1 ([3, p.22]). *Let f be a mapping with the sets A and B as domain and codomain, respectively. Then*

$$f\left(f^{-1}(T)\right) = T, \ \forall T \subseteq B \Longleftrightarrow f \text{ is onto (surjective).}$$

Note: Throughout this article, $f^{-1}(T)$ denotes the inverse image of the set T under f. The notation f^{-1} should not be confused with that of the inverse function: when f is not bijective, the inverse relation f^{-1} exists, but it is not a function.

Proposition 2.2 ([3, p.22]). *Let f be a mapping with the sets A and B as domain and codomain, respectively. Then*

$$f^{-1}(f(S)) = S, \ \forall S \subseteq A \Longleftrightarrow f \text{ is one-to-one (injective).}$$

Definition 2.3. *Let G be a group and n an integer. Then, the* nth *power function of G is defined as the function that maps each element x of G to its* nth *power x^n.*

In this article, we will discuss the power functions in the cases where $G = \mathbb{C}^*$, and where $G = \mathbb{R}^*$. In these cases, we denote the respective nth power functions as f_n and g_n, respectively:

$$f_n : \mathbb{C}^* \longrightarrow \mathbb{C}^*, \ f_n(z) = z^n, \tag{1}$$

and

$$g_n : \mathbb{R}^* \longrightarrow \mathbb{R}^*, \ g_n(x) = x^n. \tag{2}$$

Proposition 2.4 (Law of Integer Exponents, [3, Theorem 3.12]). *Let G be a group, then*

$$x^{mn} = (x^m)^n = (x^n)^m \ \ \forall x \in G, \forall m, n \in \mathbb{Z}.$$

For $G = \mathbb{C}^*$, this law can be written, in terms of the power functions f_n (1) as

$$f_{mn} = f_n \circ f_m = f_m \circ f_n \ \forall m, n \in \mathbb{Z}.$$

Proposition 2.5 ([3, Theorem 7.11]). *Let $b \in \mathbb{C}^*$ and $n \in \mathbb{N}$. Then b has exactly n distinct* nth *roots. If we write b in polar form as $b = |b| \cdot e^{i\theta}$, where $|b| > 0$ and $\theta \in \mathbb{R}$, then the* nth *roots of b are*

$$\left(\sqrt[n]{|b|}\, e^{\frac{i\theta}{n}}\right) \cdot \left(e^{\frac{2\pi i}{n}}\right)^k,$$

where $\sqrt[n]{|b|}$ denotes the principal (positive) real root, and k ranges over a complete system of residues modulo n, for example, $k = 0, 1, 2, \ldots, n-1$. Note that $e^{\frac{2\pi i}{n}}$ is an nth root of unity (that is, an nth root of 1), so that the value $(e^{\frac{2\pi i}{n}})^k$ depends only on the congruence class of k modulo n. Furthermore, in terms of the power functions (1), this translates to

$$\left|f_n^{-1}(b)\right| = n,$$

where $|A|$ for a set A denotes the cardinality of A. In particular, this implies that the fucntions f_n are onto (surjective) unless $n = 0$, but not 1-to-1 (injective) unless $n = \pm 1$.

Proposition 2.6 ([8, Theorem 4.6])**.** *Let $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ be relatively prime. If $R \subseteq \mathbb{Z}$ is a complete system of residues modulo q, then so is $pR = \{pr \,|\, r \in R\}$. In particular, by choosing $R = \{0, 1, \ldots, q-1\}$, we have that $pR = \{0, p, 2p, \ldots, (q-1)p\}$ is a complete system of residues.*

Definition 2.7. *Let* $\mathbf{F}$ *denote the set of all fractions with a natural number as denominator:*

$$\mathbf{F} = \left\{ \frac{m}{n} \,\middle|\, m \in \mathbb{Z},\, n \in \mathbb{N} \right\}.$$

Then the set of rational numbers is defined as the set of equivalence classes of fractions

$$\mathbb{Q} \cong \mathbf{F}/\sim$$

where $\sim$ denotes the usual equivalence relation defined by $\frac{m}{n} \sim \frac{m'}{n'} \overset{def}{\Longleftrightarrow} mn' = nm'$. Thus, each rational number has multiple representations by distinct but "equivalent" fractions.

The reason why we adopt this definition, allowing only positive denominators, is to avoid having to consider a radical with a negative index, like "$\sqrt[-5]{8}$", when a fraction is used as the exponent. Note however that this definition has a certain disadvantage in another context, in that the rule $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$ does not hold and has to be replaced by a more complicated $\left(\frac{a}{b}\right)^{-1} = \frac{\mathrm{sgn}(a)\cdot b}{|a|}$, where the absolute value function removes the sign from the denominator.

In this paper, the reader should be clearly aware of the distinction between a fraction and the rational number represented by the fraction: a fraction is an explicit notation like $\frac{a}{b}$, whereas a rational number is a more abstract concept defined as an equivalence class of fractions. In particular, the reader should carefully distinguish between a "fractional power" and a "rational power". We cannot properly talk about the latter until we verify that the value of the fractional power remains unchanged when the exponent is replaced by any of the equivalent fractions. How and when we can unambiguously define a rational power, indeed, is the main subject of this article.

One more item needs to be mentioned in this introduction before getting to the core of the discussion. We need to use a terminology for a "proposed definition" that has been absent from the mathematical literature. A statement is proposed as a "Conjecture" before it is afforded the title "Theorem". Similarly, we need a terminology to use for "proposed definitions / definitions in their developmental or inductive stages" while we attempt to find a valid definition. The terminology we use is "Predefinition" (see [6]).

3. Complex Number Case

Let us first consider the case where we allow the base to be a complex number. This is advantageous because, as with many situations in algebra, we get a fuller picture only when we consider the case for complex numbers. In fact, often, as is the case here, considering complex numbers simplifies the problem and makes it more transparent. In contrast, the real number case, though interesting in its own right, only gives an imperfect, partial view of the situation.

The basis of our consideration are the power functions f_n (see (1)), where n is an integer. Consider the expression b^r, where the exponent r is the reciprocal of a natural number q. In view of the laws of exponents, this means considering the "qth roots" of the complex number b. As is customary, we define a qth root of b as a complex number whose qth power equals b. Because of Proposition 2.5, the expression $b^{\frac{1}{q}}$ cannot be well-defined as a single complex number. In other words, "qth root" is not a function in the usual sense.

Still, we would like to treat "qth root" as a function on the same footing as the integral power function, and so we are led to consider both the power and root functions as set-valued functions. In doing so, to ease the notation, it will be convenient to introduce the following definitions and notations for "raising a set of numbers to a power".

Definition 3.1. *Let $S \subseteq \mathbb{C}^*$. For an integral exponent p, the set-valued power function S^p is naturally defined as giving the set of* pth *powers of the elements of S, namely*

$$S^p \stackrel{\text{def}}{=} \{b^p \mid b \in S\}.$$

This is just the image of the set S under the pth power mapping f_p:

$$S^p = f_p(S).$$

Definition 3.2. *For any set S of non-zero complex numbers and a natural number q we define*

$$S^{\frac{1}{q}} \stackrel{\text{def}}{=} \{z \in \mathbb{C}^* \mid z^q \in S\}.$$

This is equivalent to considering the inverse image of the set S under the qth power function f_q:

$$S^{\frac{1}{q}} = f_q^{-1}(S).$$

Our major interest still lies in the powers and roots of a single complex number, that is, in the case where S is a singleton set. In this case, we introduce the following abbreviated notation.

Definition 3.3. *For a non-zero complex number b and a natural number q, we apply Definition 3.2 to the singleton set $S = \{b\}$, and write*

$$b^{\frac{1}{q}} = \{b\}^{\frac{1}{q}} = \{z \in \mathbb{C}^* \mid z^q = b\} = f_q^{-1}\left(\{b\}\right).$$

In other words, we agree to interpret "b raised to the power $\frac{1}{q}$" as the set of all complex numbers whose qth power equals b, i.e. the set of qth roots of b. For example,

$$1^{\frac{1}{4}} = \{1\}^{\frac{1}{4}} = \{1, -1, i, -i\},$$

and

$$\begin{aligned} 1^{\frac{1}{3}} &= \{1\}^{\frac{1}{3}} \\ &= \{1, e^{\frac{2\pi}{3}i}, e^{-\frac{2\pi}{3}i}\} \\ &= \left\{1, -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right\} \\ &= \{1, \omega, \omega^2\}, \end{aligned}$$

where we adopt a common notation $\omega = e^{\frac{2\pi}{3}i}$, so that $e^{-\frac{2\pi}{3}i} = \omega^{-1} = \overline{\omega} = \omega^2$.

Next, we need to define the power b^r for an arbitrary rational number $r = \frac{p}{q}$. For that purpose, we will need to combine, or compose, the Definitions 3.2 of the root and 3.1 of the integral power. In light of the commutative property of multiplication, we can write $r = p \times \frac{1}{q} = \frac{1}{q} \times p$. Consequently, there are two plausible ways for composing the root and integral power functions in order to define the rational power. Namely, for a set S of nonzero complex numbers, we may try to define S^r according to either of the two conjectural definitions below, which we shall refer to as "predefinitions" because they will not be full-fledged definitions until their well-definededness is established.

Predefinition 3.4. *Let $S \subseteq \mathbb{C}^*$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $r = \frac{p}{q}$. Then S^r is defined by*

$$S^r = S^{\frac{p}{q}} = \left(S^{\frac{1}{q}}\right)^p = f_p(f_q^{-1}(S)).$$

Predefinition 3.5. *Let $S \subseteq \mathbb{C}^*$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $r = \frac{p}{q}$. Then S^r is defined by*

$$S^r = S^{\frac{p}{q}} = (S^p)^{\frac{1}{q}} = f_q^{-1}(f_p(S)).$$

Are these two predefinitions equivalent? If they are not, then is either of the two "better" than the other?

To answer these questions, the first test is that of well-definedness. Recall that a single rational number have many representations as a fraction. A good definition of rational power should not depend on such representations.

Thus the question is: if $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are two equivalent fractions, both representing the same rational number r, does each of the Predefinitions 3.4 and 3.5 above give the same result for $S^{\frac{p_1}{q_1}}$ and $S^{\frac{p_2}{q_2}}$? If under either of the predefinitions the resulting value is dependent on the fractional representation of rational numbers, then that definition is not well-defined.

As an example, the interested reader may examine $(-1)^{\frac{1}{2}}$ and $(-1)^{\frac{2}{4}}$ according to both Predefinitions 3.4 and 3.5, in order to gain insight into the difference between the two. A part of this computation is illustrated in the proof of Theorem 3.7.

The simplest example that demonstrates an answer to the above questions is the case of the first power b^1. Let n be a natural number. Since $1 = \frac{n}{n}$, we expect our definition of rational power to give $S^{\frac{n}{n}} = S^1 = S$, in particular we expect $b^{\frac{n}{n}} = \{b\}$.

First, we consider Predefinition 3.4. We have

$$\begin{aligned} b^{\frac{n}{n}} &= \{b\}^{\frac{n}{n}} = (\{b\}^{\frac{1}{n}})^n && \text{by Predefinition 3.4} \\ &= f_n(f_n^{-1}(\{b\})) = \{b\} && \text{by Proposition 2.1} \end{aligned}$$

because f_n is an onto function. More intuitively, by definition, any element of $\{b\}^{\frac{1}{n}}$ is an nth root of b, so when raised to the nth power it evaluates to b. This leads us to one of the main results of this article.

Theorem 3.6. *Suppose $\frac{p}{q}, \frac{p'}{q'} \in \mathbf{F}$ are equivalent fractions representing a rational number r, where $\frac{p}{q}$ is in the lowest terms but $\frac{p'}{q'}$ is not. Then, under Predefinition 3.4, $S^{\frac{p}{q}} = S^{\frac{p'}{q'}}$ holds for any subset $S \subseteq \mathbb{C}^*$. Thus, the general rational power given by Predefinition 3.4 is well-defined.*

Proof. Since $\frac{p'}{q'}$ is not in lowest terms, p' and q' have the greatest common divisor $d > 1$, and we have $p' = dp$ and $q' = dq$. Hence by Proposition 2.4 we have $f_{p'} = f_p \circ f_d$ and $f_{q'} = f_q \circ f_d$. Therefore

$$\begin{aligned} S^{\frac{p'}{q'}} &= f_{p'}(f_{q'}^{-1}(S)) && \text{by Predefinition 3.4} \\ &= (f_p \circ f_d)((f_q \circ f_d)^{-1}(S)) \\ &= f_p(f_d(f_d^{-1}(f_q^{-1}(S)))) \\ &= f_p(f_q^{-1}(S)) && \text{by Proposition 2.1, because } f_d \text{ is onto} \\ &= S^{\frac{p}{q}} && \text{by Predefinition 3.4} \end{aligned}$$

This establishes that we have a well-defined rational power S^r of S. □

More intuitively, in the case $S = \{b\}$, what is happening here is as follows. Since $\{b\}^{\frac{1}{q}}$ consists of q distinct qth roots of b, whose pth powers are all distinct, we have $\left|\left(\{b\}^{\frac{1}{q}}\right)^p\right| = q$. Similarly, $\{b\}^{\frac{1}{q'}}$ consists of q' distinct q'th roots of b. But some of their p'th powers are equal — in fact, q' roots are partitioned into q blocks, called cosets, each consisting of d roots whose p'th powers are equal — hence $\left|\left(\{b\}^{\frac{1}{q'}}\right)^{p'}\right| = q$ also, and in fact we have $\{b\}^{\frac{p}{q}} = \{b\}^{\frac{p'}{q'}}$.

Next, we consider Predefinition 3.5. Under that predefinition,

$$b^{\frac{n}{n}} = (\{b\}^n)^{\frac{1}{n}},$$

which consists of n distinct nth roots of b^n (one of which is b). So we have $\{b\}^{\frac{n}{n}} \neq \{b\}$, in fact $\{b\}^{\frac{n}{n}} \supsetneq \{b\}$ in this situation. The cause of this discrepancy is the fact that the power function f_n is not a one-to-one function if $|n| > 1$. Recall that by Proposition 2.2, $f^{-1}(f(S)) = S$ is true for every subset S of the domain of f (if and) only if the function f is a one-to-one function.

This example, and others, show that Predefinition 3.5 is not a "correct" way of defining rational powers. Indeed, we have the following result.

Theorem 3.7. *Suppose $\frac{p}{q}$, $\frac{p'}{q'} \in \mathbf{F}$ are equivalent fractions representing a rational number r, where $\frac{p}{q}$ is in the lowest terms but $\frac{p'}{q'}$ is not. Then, under Predefinition 3.5, $S^{\frac{p}{q}} = S^{\frac{p'}{q'}}$ does not hold in general. Thus, the general rational power as proposed by Predefinition 3.5 is not well-defined.*

Proof. Let $b \in \mathbb{C}^*$ and $S = \{b\}$. Then, when computed according to Predefinition 3.5, $\{b\}^{\frac{p}{q}}$ consists of q elements (q distinct qth roots of b^p), whereas $\{b\}^{\frac{p'}{q'}}$ consists of q' elements (q' distinct q'th roots of $b^{p'}$). Since $q' > q$, we have $\{b\}^{\frac{p}{q}} \neq \{b\}^{\frac{p'}{q'}}$, and in fact, we have $\{b\}^{\frac{p}{q}} \subsetneq \{b\}^{\frac{p'}{q'}}$. So, the rational power $\{b\}^r$ is not well-defined in this setting. □

We further illustrate this by calculating a simple example below.

Example 3.8. *Consider the base $b = -1$ and the rational exponent $\frac{1}{3} = \frac{2}{6}$. The power $(-1)^{\frac{1}{3}}$, under the computation rules of Predefinition 3.5, evaluates to*

$$\{-1\}^{\frac{1}{3}} = \left\{-1, \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i, \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right\} = \{-1, -\omega, -\omega^2\}$$

On the other hand, $\{-1\}^{\frac{2}{6}}$, under the same rules, evaluates to

$$\begin{aligned} \{-1\}^{\frac{2}{6}} &= \left(\{-1\}^2\right)^{\frac{1}{6}} \\ &= \{1\}^{\frac{1}{6}} \\ &= \left\{\pm 1, \pm\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right), \pm\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\right\} \\ &= \{\pm 1, \pm\omega, \pm\omega^2\} \end{aligned}$$

In other words, $\{-1\}^{\frac{2}{6}}$ *consists of six sixth roots of* 1*, whereas* $\{-1\}^{\frac{1}{3}}$ *consists of three cube roots of* -1 *when calculated according to Predefinition 3.5.*

Theorems 3.6 and 3.7 establish conclusively that Predefinition 3.4 is a correct definition of rational powers but not Predefinition 3.5. Thus we adopt the former as our definition:

Definition 3.9. *Let* $S \subseteq \mathbb{C}^*$*,* $p \in \mathbb{Z}$*,* $q \in \mathbb{N}$ *and* $r = \frac{p}{q}$*. Then* S^r *is defined by*

$$S^r = S^{\frac{p}{q}} = \left(S^{\frac{1}{q}}\right)^p = f_p(f_q^{-1}(S)).$$

Finally, we give a condition under which Predefinitions 3.4 and 3.5 give the same result, namely a condition under which the operations of extracting a root and raising to a power commute.

Theorem 3.10. *Let* $b \in \mathbb{C}^*$*,* $p \in \mathbb{Z}$*, and* $q \in \mathbb{N}$*. Then* $\left(b^{\frac{1}{q}}\right)^p = (b^p)^{\frac{1}{q}}$ *holds if and only if* p *and* q *are relatively prime, namely, when g.c.d.(p,q)=1.*

Proof. If p and q are not relatively prime, with the greatest common divisor d, then by Theorem 3.6, $\left(b^{\frac{1}{q}}\right)^p = \left(b^{\frac{1}{q/d}}\right)^{p/d}$ and this set consists of $\frac{q}{d}$ elements, whereas $(b^p)^{\frac{1}{q}}$ consists of q elements. Therefore $\left(b^{\frac{1}{q}}\right)^p \neq (b^p)^{\frac{1}{q}}$. This shows that the condition is necessary. To show the sufficiency, assume that p and q are relatively prime. Write b in polar form as $b = |b|\, e^{i\theta}$, with $|b| > 0$ and $\theta \in \mathbb{R}$. Then, using Proposition 2.5, we compute, on the one hand,

$$\left(b^{\frac{1}{q}}\right)^p = \left\{ \left(|b|^{\frac{p}{q}} e^{\frac{p\theta i}{q}}\right) \cdot \left(e^{\frac{2\pi i}{q}}\right)^{pk} \,\middle|\, k = 0, 1, \ldots, q-1 \right\},$$

where $|b|^{\frac{p}{q}} = \left(\sqrt[q]{|b|}\right)^p$ with the root being the principal (positive) real root. On the other hand,

$$(b^p)^{\frac{1}{q}} = \left\{ \left(|b|^{\frac{p}{q}} e^{\frac{p\theta i}{q}}\right) \cdot \left(e^{\frac{2\pi i}{q}}\right)^{k} \,\middle|\, k = 0, 1, \ldots, q-1 \right\}.$$

But since p and q are relatively prime, Proposition 2.6 implies that each of $\{pk \,|\, k = 0, 1, \ldots, q-1\}$ and $\{k \,|\, k = 0, 1, \ldots, q-1\}$ is a complete system of residues modulo q. Therefore, since the expression $(e^{\frac{2\pi i}{q}})^l$ depends only on the congruence class of the exponent l modulo q, the two sets $\left\{ \left(e^{\frac{2\pi i}{q}}\right)^{pk} \,\middle|\, k = 0, 1, \ldots, q-1 \right\}$ and $\left\{ \left(e^{\frac{2\pi i}{q}}\right)^{k} \,\middle|\, k = 0, 1, \ldots, q-1 \right\}$ are equal, and hence $\left(b^{\frac{1}{q}}\right)^p$ and $(b^p)^{\frac{1}{q}}$ are equal. □

4. Real Number Case

Next we consider the real number case. We consider real base b and only the real values of its power b^r, for a rational exponent r. Here, it will turn out that even the "good" definition given as Definition 3.9 does not work unconditionally.

The basis of our consideration is again the power function g_n (2), but now with both domain and codomain restricted to be the set of non-zero real numbers $\mathbb{R}^*$. As in the complex case, and keeping the same notation, we consider powers and roots as set-valued functions. Namely, we will give the following definitions.

Definition 4.1. *Let $S \subseteq \mathbb{R}^*$, $p \in \mathbb{Z}$. Then*

$$S^p = \{x^p \,|\, x \in S\} = g_p(S).$$

Definition 4.2. *Let $S \subseteq \mathbb{R}^*$, $q \in \mathbb{N}$. Then*

$$S^{\frac{1}{q}} = \{x \in \mathbb{R}^* \,|\, x^q \in S\} = g_q^{-1}(S).$$

Definition 4.3. *For a non-zero real number b and a natural number q, we write*

$$b^{\frac{1}{q}} = \{b\}^{\frac{1}{q}} = \{x \in \mathbb{R}^* \,|\, x^q = b\} = g_q^{-1}(\{b\}).$$

Then we have:

- If q is even, then

$$b^{\frac{1}{q}} = \{b\}^{\frac{1}{q}} = \begin{cases} \{\sqrt[q]{b}, -\sqrt[q]{b}\} & \text{if } b > 0 \\ \emptyset & \text{if } b < 0 \end{cases}.$$

- If q is odd, then

$$b^{\frac{1}{q}} = \{b\}^{\frac{1}{q}} = \{\sqrt[q]{b}\}.$$

The case of an odd q is simpler than that of an even q. In fact when q is odd we have the single-valued qth root. This is because the qth power mapping g_q from $\mathbb{R}^*$ to $\mathbb{R}^*$ is one-to-one and onto (bijective) if q is odd, whereas g_q is neither one-to-one nor onto if q is even.

For the general fractional power $b^{\frac{p}{q}}$, let us examine the two Predefinitions 3.4 (which we have adopted as Definition 3.9) and 3.5 (which we have discarded as not well-defined) that we studied in the complex case, but restricted to real numbers (in other words, with f_p and f_q replaced by g_p and g_q, respectively). As before, we shall refer to these conjectural definitions simply as predefinitions.

Predefinition 4.4. *Let $S \subseteq \mathbb{R}^*$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $r = \frac{p}{q}$. Then S^r is defined by*

$$S^r = S^{\frac{p}{q}} \;=\; (S^{\frac{1}{q}})^p \;=\; g_p(g_q^{-1}(S))$$

Predefinition 4.5. *Let $S \subseteq \mathbb{R}^*$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $r = \frac{p}{q}$. Then S^r is defined by*

$$S^r = S^{\frac{p}{q}} \;=\; (S^p)^{\frac{1}{q}} \;=\; g_q^{-1}(g_p(S))$$

Let's first look at the "problematic" Predefinition 4.5.

Example 4.6. *Let $r = \frac{3}{2}$. Then r can also be written as $\frac{6}{4}$. Then on the one hand*

$$\{-1\}^{\frac{3}{2}} = (\{-1\}^3)^{\frac{1}{2}} = \{-1\}^{\frac{1}{2}} = \emptyset,$$

but on the other hand

$$\{-1\}^{\frac{6}{4}} = (\{-1\}^6)^{\frac{1}{4}} = \{1\}^{\frac{1}{4}} = \{\pm 1\}.$$

So, we have $\{-1\}^{\frac{3}{2}} \neq \{-1\}^{\frac{6}{4}}$.

Thus, as expected, rational power is not well-defined under the Predefinition 4.5.

Next, let us try Predefinition 4.4, which is analogous to the "good" definition given in Section 3 (Definition 3.9) for complex numbers.

Example 4.7. *Let $r = \frac{3}{2} = \frac{6}{4}$. Then*

$$\begin{aligned} \{1\}^{\frac{3}{2}} &= (\{1\}^{\frac{1}{2}})^3 = \{\pm 1\}^3 = \{\pm 1\}, \\ \{1\}^{\frac{6}{4}} &= (\{1\}^{\frac{1}{4}})^6 = \{\pm 1\}^6 = \{1\}, \end{aligned}$$

so that $\{1\}^{\frac{3}{2}} \neq \{1\}^{\frac{6}{4}}$.

This example establishes that even our "good" Predefinition 4.4 is not well-defined in the real number case, that is, it does not define rational power b^r properly within the set of real numbers.

In order to salvage the situation, what we must do is to insist on writing the fractional exponent in the lowest terms. In other words, for a nonzero real number b and a rational number r, we shall define the real-valued power b^r as follows.

Definition 4.8. *Let b be a non-zero real number and r be a rational number. Write $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, p and q are relatively prime (i.e. consider the standard / lowest-terms repesentative of r). Then we define the real-valued power b^r by*

$$b^r = \left(\sqrt[q]{b}\right)^p,$$

where $\sqrt[q]{b}$ is taken to be the principal root, provided it exists as a real number.

Under this definition, we have the following familiar result.

Theorem 4.9. *Let b be a non-zero real number and r be a rational number. Write $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, p and q are relatively prime. Then, under Definition 4.8, we have,*

- *If q is even (so p is necessarily odd) then*

$$\{b\}^{\frac{p}{q}} = \begin{cases} \left\{ \left(\sqrt[q]{b}\right)^p, \ -\left(\sqrt[q]{b}\right)^p \right\} & \text{if } b > 0 \\ \emptyset & \text{if } b < 0 \end{cases}$$

- *if q is odd (p can be either odd or even) then*

$$\{b\}^{\frac{p}{q}} = \left\{ \left(\sqrt[q]{b}\right)^p \right\}$$

Proof. The proof is familiar and is not included here. □

We have seen that the operations of raising to a power and extracting a root within real numbers do not commute in general. The following theorem determines when it will be safe to commute between the two operations.

Theorem 4.10. *Let $p \in \mathbb{Z}$, and $q \in \mathbb{N}$. Then the relation $\left(S^{\frac{1}{q}}\right)^p = (S^p)^{\frac{1}{q}}$ holds for any subset $S \subseteq \mathbb{R}^*$ if and only if at least one of p or q is odd, namely, when g.c.d.(p,q,2)=1. In other words, in the real number case the operations of extracting a root and raising to a power do not commute if and only both p and q are even. In particular, they commute if the fraction $\frac{p}{q}$ is in lowest terms.*

Proof. The equality in question can be written as

$$g_p(g_q^{-1}(S)) = g_q^{-1}(g_p(S)).$$

The key facts used in the proof are the following two.

(1) When n is an odd integer, the nth power mapping g_n is bijective. Hence, it has the inverse mapping g_n^{-1} (the nth root mapping) that satisfies $g_n \circ g_n^{-1} = g_n^{-1} \circ g_n = id$, where id denotes the identity mapping of $\mathbb{R}^*$.
(2) For two integers n and m, the nth power mapping g_n and mth power mapping g_m commute: $g_n \circ g_m = g_m \circ g_n$ (see Proposition 2.4).

First suppose that q is odd. Then g_q is bijective, hence it has the inverse mapping g_q^{-1}, and we compute

$$\begin{aligned} g_p(g_q^{-1}(S)) &= (g_q^{-1} \circ g_q)(g_p(g_q^{-1}(S))) && \text{Since } g_q^{-1} \circ g_q = id \\ &= g_q^{-1}((g_q \circ g_p)(g_q^{-1}(S))) \\ &= g_q^{-1}((g_p \circ g_q)(g_q^{-1}(S))) && \text{Since } g_q \circ g_p = g_p \circ g_q \\ &= g_q^{-1}(g_p((g_q \circ g_q^{-1})(S))) \\ &= g_q^{-1}(g_p(S)) && \text{Since } g_q \circ g_q^{-1} = id \end{aligned}$$

Next suppose that p is odd. Then g_p is bijective, hence it has the inverse mapping g_p^{-1}, and we compute

$$\begin{aligned} g_p(g_q^{-1}(S)) &= g_p(g_q^{-1}((g_p^{-1}\circ g_p)(S))) && \text{Since } g_p^{-1}\circ g_p = id \\ &= g_p(g_q^{-1}(g_p^{-1}(g_p(S)))) \\ &= g_p(g_p^{-1}(g_q^{-1}(g_p(S)))) && \text{Since } g_p\circ g_q = g_q\circ g_p \\ &= (g_p\circ g_p^{-1})(g_q^{-1}(g_p(S))) \\ &= g_q^{-1}(g_p(S)) && \text{Since } g_p\circ g_p^{-1} = id \end{aligned}$$

Finally, suppose p and q are both even, and let $S = \{b\}$ with $b \in \mathbb{R}^*$. If $b > 0$, then $\left(\{b\}^{\frac{1}{q}}\right)^p = \left\{\left(\sqrt[q]{b}\right)^p\right\}$, whereas $(\{b\}^p)^{\frac{1}{q}} = \left\{\left(\sqrt[q]{b}\right)^p, -\left(\sqrt[q]{b}\right)^p\right\}$. If $b < 0$, then $\left(\{b\}^{\frac{1}{q}}\right)^p = \emptyset$, whereas $(\{b\}^p)^{\frac{1}{q}} = \left\{\sqrt[q]{b^p}, -\sqrt[q]{b^p}\right\}$. Hence, if p and q are both even, then $\left(\{b\}^{\frac{1}{q}}\right)^p \neq (\{b\}^p)^{\frac{1}{q}}$. □

We have seen that in the real number case, even with the supposedly good definition given as Predefinition 4.4, we must insist that the exponent be in the lowest terms in order to define a rational power (as in Definition 4.8), and Theorem 4.10 says that under that restriction of the exponent, either of the Predefinitions 4.4 or 4.5 coincides with Definition 4.8. Thus it may look as if it does not matter which of the predefinitions we adopt as far as we are dealing only with real numbers. However, as we showed in the previous section, Predefinition 4.4 is the one that generalizes well to the complex number case, hence it is inherently better than Predefinition 4.5. Therefore, for pedagogical reasons, even in the real case the instruction should give a clear preference to the "good" definition given by Predefinition 4.4 (as incorporated into Definition 4.8), in order to better prepare students for future more advanced mathematical experiences, for instance when they may study complex numbers.

5. Conclusion: Implications to Teaching

In this section, we abandon our "set-function" notation for powers and roots, and shall discuss the usual convention under which $\sqrt[q]{b} = b^{\frac{1}{q}}$ denotes the principal qth root of b. Then we will make suggestions as to how textbooks and classroom teaching should treat powers with fractional exponents. Recall that the principal root means the following: if $b > 0$, then $\sqrt[q]{b}$ denotes the positive qth root of b; if $b < 0$ and q is odd, it denotes the negative qth root of b; and if $b < 0$ and q is even, then $\sqrt[q]{b}$ is "undefined" (as a real number). When complex numbers are included in consideration, the square root $\sqrt{b}$, with $b < 0$, is often understood to mean the purely imaginary number $\sqrt{|b|} \cdot i$. We shall also refer to this convention in the discussion.

In many high-school or beginning college level textbooks, two computation rules are given as definitions of rational powers (of real numbers). One is

$$b^{\frac{p}{q}} = \left(\sqrt[q]{b}\right)^p \tag{A}$$

and the other is

$$b^{\frac{p}{q}} = \sqrt[q]{b^p}. \tag{B}$$

Note that the rule (A) corresponds to our Predefinitions 3.4 and 4.4 (which we have adopted as Definitions 3.9 and 4.8), and the rule (B) corresponds to Predefinitions 3.5 and 4.5 (which we discarded because they are not well-defined). Usually textbooks present both of these two equalities, often written side by side, giving the novice an impression that the two expressions on the right hand sides are equivalent (reference instances are many and we can not be exhaustive, we cite a small random sampling without any implication: [2, 4, 7, 9]). But as we have clarified, these two rules are not equivalent, (A) is the "correct" one, and (B) is not.

There may be several reasons why the two rules are often confused. The root cause of the problem we believe is the lack of awareness of the order of operations. Students often are not clearly aware that the order of operations matters, and implicitly assume the commutativity of order of operations in many cases (after all our instructions have emphasized and reemphasized it for them). Commutativity holds for multiplication of numbers, as in $\frac{1}{q} \cdot p = p \cdot \frac{1}{q}$, but it does not translate to the commutativity of the operations of raising to a power and of extracting roots.

In fact, there is a tendency that between the rules (A) and (B) the "wrong" rule (B) is not only accepted but actually preferred, especially by the students. One, linguistic, reason for this may be that in the English language, when one reads a fraction the numerator is read first and then the denominator. Thus it may be more "natural", when people see the expression $b^{\frac{p}{q}}$, to raise b to the pth power first, then take the qth root. Another reason may be typographical. The notation $\sqrt[q]{b^p}$ is easier to read and write than $\left(\sqrt[q]{b}\right)^p$, which requires parentheses. But these reasons are linguistic or notational accidents, and should in no way take precedence over mathematical correctness.

On the other hand, there are also factors that lend support to the "correct" rule (A). One is the fact that rule (A) often results in more efficient numerical computation of the value of the power, as is often pointed out in textbooks. For example, to compute $8^{\frac{5}{3}}$, if one uses rule (B), as many students would do, one has to first compute 8^5, which perhaps requires a calculator to do it quickly. But if one uses rule (A), then one first finds the cube root 2 of 8, and then raises it to the 5th power to get the result 32. This can easily be done by mental arithmetic.

Still, computational efficiency is not the central reason why rule (A) should be favored. Rather, the most fundamental issue in deciding which of the two rules should be adopted is a logical one, namely that of well-definedness: whenever we use fractional expressions, we expect that two expressions $\frac{a}{b}$ and $\frac{a\cdot c}{b\cdot c}$ to be equivalent and yield same results. Predefinition 3.4, which is an extension of rule (A) to the complex numbers, meets this expectation, but Predefinition 3.5, which is an extension of rule (B) to complex numbers, violates it, as we have seen in Section 3. Therefore, Predefinition 3.4 is logically sound, whereas 3.5 is logically deficient as it does not define rational powers, only fractional powers.

A simple illustration of the contrast between the two is given by the equality

$$b^{\frac{n}{n}} = b^{\frac{1}{1}} = b \tag{3}$$

which should be an identity if the definition of the power is a good one. Under rule (A), the equality (3) translates to $\left(\sqrt[n]{b}\right)^n = b$, which is always true (whenever $\sqrt[n]{b}$ exists) by the very definition of the nth root $\sqrt[n]{b}$ of b as a number whose nth power equals b. On the other hand, under rule (B), the same equality (3) translates to $\sqrt[n]{b^n} = b$, which is not guaranteed to be true, because even though b is by definition an nth root of b^n, there are several other nth roots of b^n. In fact, if we consider complex numbers, $(b^n)^{\frac{1}{n}}$ would be n-valued, and there is no canonical way of singling out any one of the n values as the "principal" value of the root, either. Therefore, the expression $(b^n)^{\frac{1}{n}}$ is not well-defined if it were to represent a single value.

The simplest case of the above is the equality

$$(-1)^{\frac{2}{2}} = -1. \tag{4}$$

We follow the convention under which $\sqrt{b}$ denotes $\sqrt{|b|}\cdot i$ when $b < 0$. With rule (A) we have

$$\left(\sqrt{-1}\right)^2 = i^2 = -1$$

hence the equality (4) holds. On the other hand, with rule (B), we have

$$\sqrt{(-1)^2} = \sqrt{1} = +1$$

hence the equality (4) does not hold. This shows that, against our naive expectation, the operations of squaring and extracting square root do not commute. Correspondingly, in our set-function notation, we have a contrast between

$$\left(\{-1\}^{\frac{1}{2}}\right)^2 = \{i, -i\}^2 = \{i^2, (-i)^2\} = \{-1, -1\} = \{-1\}$$

under the "good" Predefinition 3.4, and

$$\left(\{-1\}^2\right)^{\frac{1}{2}} = \{1\}^{\frac{1}{2}} = \{1, -1\} \neq \{-1\}$$

under the "problematic" Predefinition 3.5.

On the basis of the above, our suggestions to textbook authors and mathematics instructors regarding rational and fractional powers are the following.

- When introducing the fractional power $b^{\frac{p}{q}}$, use only the correct rule $b^{\frac{p}{q}} = \left(\sqrt[q]{b}\right)^p$. We should not only avoid using the incorrect rule $b^{\frac{p}{q}} = \sqrt[q]{b^p}$ but provide counterexamples to demonstrate its invalidity. In other words, emphasize that the operations of extracting a root and raising to a power do not commute. A conspicuous case of non-commutativity like this can be used as a lesson against errors caused by careless assumptions of commutativity be it in the order of operations or for noncommutative binary operations. Note that incorrect assumptions about commutativity of order of operations is a source of many common student misconceptions and errors. Awareness of these issues will prepare students to accept non-commutativity of operations (for example, division, subtraction, matrix multiplication, and function composition) as essential part of their mathematical experience.
- When dealing only with real numbers, one may present examples where the value of $b^{\frac{p}{q}}$ changes when the exponent is replaced by another equivalent fraction, even under the "correct" rule. But assure the students that this discrepancy will be resolved with the introduction of complex numbers, hence that the "correct" rule is indeed the correct one!

One possible counterargument to our suggestions may be that if we limit ourselves to positive real numbers, both rules work fine and give the same result, and therefore it is alright to treat both $\sqrt[q]{b^p}$ and $\left(\sqrt[q]{b}\right)^p$ as well-defined and equivalent to each other. But that is dangerous, for the following two reasons:

- The human mind has the tendency to reduce definitions and theorems to formulae. Namely, a component of the definition or the theorem becomes a "formula" and other parts would be treated as auxiliary and easily forgotten. For instance, once $b^{\frac{p}{q}} = \sqrt[q]{b^p}$ is called a formula, it becomes the main part of the theorem and everything else, such as the restriction that b should be a positive real number, becomes auxiliary.
- Habit of mind, once formed, is hard to correct.

Thus, students who are taught in such a way that deemphasizes the distinction between $\left(\sqrt[q]{b}\right)^p$ and $\sqrt[q]{b^p}$ are more likely to:

- have misconceptions similar to that given in the introduction to this article,
- have difficulty adjusting to new situations that arise with complex numbers,

- have the misconception that the operations of extracting roots and raising to a power commute.

We also have the following suggestions regarding the treatment of powers and radicals in general:

- When only real numbers are considered, the radicals exhibit sharply contrasting properties for even and odd indices. This fact should be treated in relation to the properties of corresponding power functions: an odd power function is bijective (both one-to-one and onto) whereas an even power function is neither one-to-one nor onto. These different properties of the power functions account for the different behaviors of radicals with odd and even indices. Such functional viewpoint will prepare a common context in which to discuss power and roots of complex numbers in subsequent courses. (In elementary courses, the concepts of one-to-one and onto functions can be illustrated informally using the graphs of the power functions, if one wishes to avoid going into their technical definitions, cf. [5].)
- When complex numbers are introduced, the new behaviors of the radicals (roots) can similarly be understood in terms of the properties of corresponding power functions. Complex power functions, in contrast to the real ones, are onto but not one-to-one, regardless of whether the exponent is even or odd. As a consequence, in the complex case the radicals with even and odd indices behave similarly to each other (in other words, the distinction seen in the real case disappears).

There are at least two underlying general principles that we advocate and want to emphasize. First, we should try to minimize, or better yet to avoid, reduction of definitions and theorems into formulae. (It should be noted that reduction of definitions and theorems into formulae results in oversimplification and truncation. An effective way to help students avoid reducing definitions and theorems into formulae is for instructors to provide sufficient repetition of and to require students to cite all components of definitions and theorems when using the formulaic components until the whole concept becomes part of students' schema.) Second, we should try to minimize the adjustment students are required to make when they advance to encounter more general situations of a given topic. In the case of the definition of rational and fractional powers, it is thus much better, as we suggested, to insist from the beginning on using correct definitions that do not require modifications afterwards. It is much more efficient and hence better not to have to force students to unlearn what they have learned with real numbers when they encounter complex numbers! As another example of the same principle, the functional viewpoint that we suggest gives a unifying context in which to understand the seemingly quite different behaviors of radicals in the real and complex cases, and thus will ease the transition between the two topics for the students. When teaching elementary courses, we must take care, as suggested in this note, so that we may prepare the students better both for their current and future studies and for their life experiences.

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