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A SOLUTION FOR THE RHIND PAPYRUS UNIT FRACTION DECOMPOSITIONS

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Abstract. The long unsolved mystery of a method for constructing the unit fraction decomposition table in the Rhind Papyrus now has a systematic solution that is consistent with the mathematical processes used in the Rhind Papyrus. The solution within this paper is the result of work done during the fall 2006.

1. Introduction

Most of our knowledge of ancient Egyptian mathematics is derived from two sizable papyri - the Rhind Papyrus and the Golenischev Papyrus. In 1858 A. Henry Rhind purchased a partial papyrus in Luxor, Egypt. The papyrus was reportedly found in Thebes, in the ruins of a building near the Ramesseum. The Rhind Papyrus was written in hieratic script about 1650 B. C. by a scibe named Ahmes. Since much of what we know about ancient Egyptian mathematics comes from the Rhind Papyrus, Ahmes' work is mathematically and historically significant.

Readers of the papyrus are assured that its content is a likeness of earlier work dating back to the Twelfth Dynasty, 1849-1801 B. C. Early Egyptian mathematics with fractions, dating back to the Twelfth Dynasty, was made difficult by the computational practice of allowing only unit fractions, i.e., fractions of the form $\frac{1}{n}$, where n is a natural number. To overcome the difficulty, the ancient Egyptian mathematicians constructed unit fraction decomposition tables for quick reference and use. The table at the beginning of the Rhind Papyrus gives unit fraction decompositions of fractions of the form $\frac{2}{n}$, where n is an odd natural number between 3 and 103, and is the most extensive of the unit fraction decomposition tables to be found and preserved among the ancient Egyptian papyri.

As stated within the Rhind Papyrus, the content of the papyrus was to give a thorough study of all things, insight into all that exists, and knowledge of all obscure secrets". However, no insight or knowledge of the construction of the unit fraction decomposition table was given within the papyrus, leaving its method of construction an obscure secret. Through the many years, interested people have tried to solve the mystery of the construction of the Rhind Papyrus unit fraction decomposition table. Because of the insights and systematic processes that are included within the Rhind Papyrus, most of the interested people looked for a systematic process that would give the unit fraction decompositions, but up to the fall 2006, only patterns within the table were known. For example, it was

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known that fractions within the table of the form $\frac{2}{3k}$ follow the general pattern $\frac{2}{3k} = \frac{1}{2k} + \frac{1}{6k}$. Prior to the fall 2006, there were unit fraction decompositions that followed none of the known patterns. I became aware of the long unsolved mystery during early fall 2006, while teaching a mathematics history class. Given below is a systematic solution consistent with the content of the Rhind Papyrus obtained during the fall 2006 that gives all the unit fraction decompositions.

Much of the information above is included in David Burton's book *The History of Mathematics An Introduction*[1]. Additional information about the Rhind Papyrus and known patterns for entries within the the Rhind Papyrus unit fraction decomposition table are available in Burton's book.

2. The Solution

The ideal decomposition for a fraction within the Rhind Papyrus arithmetic table would be the sum of two unit fractions with small denominators for best use of already constructed table entries and that serves most efficiently as a reference for additional calculations. For a fraction $\frac{2}{n}$, the obvious place to start would be to rewrite $\frac{2}{n}$ in the form $\frac{n+1}{np}$, where p was the appropriate natural number, and gives the unit fraction decomposition $\frac{1}{p} + \frac{1}{np}$. There is, in fact, historical support for such a use of "$n + 1$".

The ancient Egyptian division process gave the unit fraction decomposition $\frac{1}{2}+\frac{1}{6}$ for $\frac{2}{3}$. Using knowledge known by the ancient Egyptian mathematicians, $\frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{3+1}{6}$ suggesting that the use of "$n + 1$" could be helpful when looking for efficient unit fraction decompositions. In addition, the use of $n + 1$ gives the unit fraction decompositions at the beginning of the table: $\frac{2}{5} = \frac{5+1}{3\times5} = \frac{3\times2}{3\times5} = \frac{1}{3} + \frac{1}{15}$ and $\frac{2}{7} = \frac{7+1}{4\times7} = \frac{1}{4} + \frac{1}{28}$. However, the use of 9+1 does not give an efficient unit fraction decomposition for $\frac{2}{9}$ as seen by $\frac{4}{9} = 2 \times \frac{2}{9} = \frac{2}{5} + \frac{2}{45}$, where the needed unit fraction decomposition for $\frac{2}{45}$ was unknown and would remain unknown until much later in the table. Thus there was a need to adapt the additive process, which led to the used of 9+o, where o is an odd natural number between 1 and n. If, in fact, the creator of the unit fraction decompositions used the $9 + o$, trial and error could have been used to determine the "best" replacement for o. As an alternative, the creator could have used the information below to more quickly select the "best" replacement for o.

For $\frac{2}{n} = \frac{n+o}{np} = \frac{1}{p} + \frac{o}{np} = \frac{2p}{np}$ to be a two term decomposition, o would have to divide np. If o divides n, then o is a product of prime factors of n, where repetitions could occur and the product is less than n. Thus consider the case that o does not divide n. Let s be a prime factor of o. Since o is odd, s is odd. Let u be the natural number such that $o = us$. Since s divides o and o divides np, then s divides n or s divides p. If s divides p, there exists a natural number k such that $p = ks$, in which case $n = 2p - o = 2ks - us = (2k - u)s$ and s divides n. Hence s divides n. Since s divides n, $n + o = n + us = 2p$, and s does not divide 2, then s divides p. Hence s divides both n and p and o is a product of two or more prime numbers that divide both n and p, where repetitions could occur, the product is less than n, the product divides np, and the product does not divide n. The value of o that best satisfies the objectives is selected.

Application of the above process gives the following table entries: $\frac{2}{9} = \frac{2}{3\times3} = \frac{9+3}{6\times9} = \frac{3+1}{2\times9} = \frac{2\times2}{2\times9} = \frac{1}{6} + \frac{1}{18}$; $\frac{2}{15} = \frac{2}{3\times5} = \frac{15+5}{10\times15} = \frac{3+1}{2\times15} = \frac{2\times2}{3\times15} = \frac{1}{10} + \frac{1}{30}$;

$\frac{2}{21} = \frac{2}{3\times 7} = \frac{21+7}{14\times 21} = \frac{3+1}{2\times 21} = \frac{2\times 2}{2\times 21} = \frac{1}{14} + \frac{1}{42}$; $\frac{2}{27} = \frac{2}{3\times 3\times 3} = \frac{27+9}{18\times 27} = \frac{3+1}{2\times 27} = \frac{2\times 2}{2\times 27} = \frac{1}{18} + \frac{1}{54}$; and $\frac{2}{33} = \frac{2}{3\times 11} = \frac{33+11}{22\times 33} = \frac{3+1}{2\times 33} = \frac{2\times 2}{2\times 33} = \frac{1}{22} + \frac{1}{66}$. The pattern above continues for fractions of the form $\frac{2}{3k}$ giving $\frac{2}{3k} = \frac{2\times 2}{2\times 3k} = \frac{3+1}{2\times 3k} = \frac{1}{2k} + \frac{1}{6k}$, as cited in Burton's book.

Moving onward, the process gives $\frac{2}{25} = \frac{2}{5\times 5} = \frac{25+5}{15\times 25} = \frac{5+1}{3\times 25} = \frac{1}{15} + \frac{1}{75}$; $\frac{2}{35} = \frac{2}{5\times 7} = \frac{35+25}{30\times 35} = \frac{1}{30} + \frac{1}{42}$; $\frac{2}{49} = \frac{2}{7\times 7} = \frac{49+7}{28\times 49} = \frac{1}{28} + \frac{1}{306}$; $\frac{2}{55} = \frac{2}{5\times 11} = \frac{55+5}{30\times 55} = \frac{1}{30} + \frac{1}{330}$; $\frac{2}{65} = \frac{2}{5\times 13} = \frac{65+13}{39\times 65} = \frac{1}{39} + \frac{1}{195}$; $\frac{2}{77} = \frac{2}{7\times 11} = \frac{77+11}{44\times 77} = \frac{1}{44} + \frac{1}{308}$; $\frac{2}{85} = \frac{2}{5\times 17} = \frac{85+17}{51\times 85} = \frac{1}{51} + \frac{1}{255}$; and $\frac{2}{91} = \frac{2}{7\times 13} = \frac{91+49}{70\times 91} = \frac{1}{70} + \frac{1}{130}$; and $\frac{2}{95} = \frac{2}{5\times 19} = \frac{95+25}{60\times 95} = \frac{1}{60} + \frac{1}{228}$.

The remaining entries in the table are of the form $\frac{2}{n}$, where n is prime and greater than 7. For consistency and the desire for a two fraction decomposition of $\frac{2}{n}$, start with n+1. If n+1 does not give an efficient decomposition of $\frac{2}{n}$, then, as above, select an odd integer $u > 1$ and generate p where $\frac{2}{n} = \frac{n+u}{np} = \frac{2p}{np}$. If u divides p, then u would be a candidate for the odd integer o added to n to obtain the most efficient decomposition for $\frac{2}{n}$. If u does not divide p, then for $\frac{u}{np}$ to reduce to unit fractions, u would have to be written as a sum of decreasing integers all of which are factors of p. Thus the choice for o would be an odd integer that divides p and/or can be written as a sum of decreasing numbers all of which divide p, where for a fixed o, the sum with largest last term would be preferred.

The above technique, when applied to the decomposition for $\frac{2}{95}$ given above, gives the more efficient form $\frac{2}{95} = \frac{2}{5\times 19} = \frac{95+25}{60\times 95} = \frac{1}{60} + \frac{25}{60\times 95} = \frac{1}{60} + \frac{15+10}{60\times 95} = \frac{1}{60} + \frac{15}{60\times 95} + \frac{10}{60\times 95} = \frac{1}{60} + \frac{1}{380} + \frac{1}{570}$ found in the Rhind Papyrus.

Application of the process gives $\frac{2}{11} = \frac{11+1}{6\times 11} = \frac{1}{6} + \frac{1}{66}$. However, the use of $13+1$ does not provide an efficient decomposition of $\frac{2}{13}$ and the search for the desired odd integer o greater than 1 would begin. One method for obtaining o would be trial and error using odd numbers starting with 3 and going to 11, which would be, at best, tedious; making another method highly desirable.

Let n be prime and sufficiently large. Because for each odd integer $2k+1$ between 1 and n, $\frac{2k+1}{n} = \frac{2k}{n} + \frac{1}{n}$, a decomposition of $\frac{2}{n}$ that efficiently gives $\frac{2k}{n}$ also efficiently gives $\frac{2k+1}{n}$, making the fractions $\frac{2k}{n}$ the focus of attention for efficiency in calculations. Because of the pattern of behavior of fractions of the form $\frac{2}{3k}$, for efficient calculation of $\frac{4}{n} = 2(\frac{2}{n})$, the number p would need a factor of 2 or a factor of 3. For efficient calculation of $\frac{6}{n} = 3(\frac{2}{n})$, p would continue to need a factor of 2 or 3. For efficient calculation of $\frac{8}{n} = 4(\frac{2}{n})$, p would need a factor of 3 or 4. For efficient calculation of $\frac{10}{n} = 5(\frac{2}{n})$, p would need a factor of 4 or 6 or 15 starting with 6. For efficient calculation of $\frac{12}{n} = 6(\frac{2}{n})$, p would need a factor of 4 or 6 or 15 starting with 8. For efficient calculation of $\frac{2k}{n}$, where k is greater than or equal to 7, p would need a factor of 4 or 6 starting with 12. For the entries in the table, the maximum value of p is 100. Those multiple of 4 or 6 within the limit that have a factor of 11 or 13 or 17 or 19 or 23 will be less efficient in calculations than other suitable choices and can be eliminated from consideration.

Using the information above, for $\frac{2}{13}$, the considerations for p are 8 or 12, which gives $\frac{2}{13} = \frac{13+3}{8\times 13} = \frac{13+2+1}{8\times 13} = \frac{1}{8} + \frac{1}{52} + \frac{1}{104}$. For $\frac{2}{17}$, the considerations for p are 12 or 16, which gives $\frac{2}{17} = \frac{17+7}{12\times 17} = \frac{17+4+3}{12\times 17} = \frac{1}{12} + \frac{1}{51} + \frac{1}{68}$. For $\frac{2}{19}$, $20 \leq 2p < 38$ and $10 \leq p < 19$, giving $p = 12, 16, or 18$ and $\frac{2}{19} = \frac{19+5}{12\times 19} = \frac{19+3+2}{12\times 19} = \frac{1}{12} + \frac{1}{76} + \frac{1}{114}$.

Continued application of the process gives $\frac{2}{23} = \frac{23+1}{12\times 23} = \frac{1}{12} + \frac{1}{276}$, $\frac{2}{29} = \frac{29+19}{24\times 29} = \frac{29+12+4+3}{24\times 29} = \frac{1}{24} + \frac{1}{58} + \frac{1}{174} + \frac{1}{232}$, $\frac{2}{31} = \frac{31+9}{20\times 31} = \frac{31+5+4}{20\times 31} = \frac{1}{20} + \frac{1}{124} + \frac{1}{155}$, $\frac{2}{37} = \frac{37+11}{24\times 37} = \frac{37+8+3}{24\times 37} = \frac{1}{24} + \frac{1}{111} + \frac{1}{296}$, and $\frac{2}{41} = \frac{41+7}{24\times 41} = \frac{41+4+3}{24\times 41} = \frac{1}{24} + \frac{1}{246} + \frac{1}{328}$, $\frac{2}{43} = \frac{43+41}{42\times 43} = \frac{43+21+14+6}{42\times 43} = \frac{1}{42} + \frac{1}{86} + \frac{1}{129} + \frac{1}{301}$, $\frac{2}{47} = \frac{47+13}{30\times 47} = \frac{47+10+3}{30\times 47} = \frac{1}{30} + \frac{1}{141} + \frac{1}{470}$, $\frac{2}{53} = \frac{53+7}{30\times 53} = \frac{53+5+2}{30\times 53} = \frac{1}{30} + \frac{1}{318} + \frac{1}{795}$, $\frac{2}{59} = \frac{59+13}{36\times 59} = \frac{59+9+4}{36\times 59} = \frac{1}{36} + \frac{1}{236} + \frac{1}{531}$, $\frac{2}{61} = \frac{61+19}{40\times 61} = \frac{61+10+5+4}{40\times 61} = \frac{1}{40} + \frac{1}{244} + \frac{1}{610}$, $\frac{2}{67} = \frac{67+13}{40\times 67} = \frac{67+8+5}{40\times 67} = \frac{1}{40} + \frac{1}{335} + \frac{1}{536}$, $\frac{2}{71} = \frac{71+9}{40\times 71} = \frac{71+5+4}{40\times 71} = \frac{1}{40} + \frac{1}{568} + \frac{1}{710}$, $\frac{2}{73} = \frac{73+47}{60\times 73} = \frac{73+20+15+12}{40\times 73} = \frac{1}{40} + \frac{1}{219} + \frac{1}{292} + \frac{1}{365}$, $\frac{2}{79} = \frac{79+41}{60\times 79} = \frac{79+20+15+6}{60\times 79} = \frac{1}{60} + \frac{1}{237} + \frac{1}{316} + \frac{1}{790}$, $\frac{2}{83} = \frac{83+37}{60\times 83} = \frac{83+15+12+10}{60\times 83} = \frac{1}{60} + \frac{1}{332} + \frac{1}{415} + \frac{1}{498}$, $\frac{2}{89} = \frac{89+31}{60\times 89} = \frac{89+15+10+6}{60\times 89} = \frac{1}{60} + \frac{1}{356} + \frac{1}{534} + \frac{1}{890}$, and $\frac{2}{97} = \frac{97+15}{56\times 97} = \frac{97+8+7}{56\times 97} = \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$.

Applying the process to the last entry within the table gives $\frac{2}{101} = \frac{101+11}{56\times 101} = \frac{101+7+4}{56\times 101} = \frac{1}{56} + \frac{1}{808} + \frac{1}{1414}$, which is not efficient for additional calculations. To obtain a more efficient decomposition the method above is adapted in a very natural manner as follows to obtain the last entry within the table: $\frac{2}{101} = \frac{1}{101} + \frac{6}{6\times 101} = \frac{1}{101} + \frac{3+2+1}{6\times 101} = \frac{1}{101} + \frac{1}{202} + \frac{1}{303} + \frac{1}{601}$.

As exhibited in the Rhind Papyrus, the ancient Egyptian mathematicians cleverly used "addition" to obtain systematic processes for accomplishing their mathematical objectives. Thus the use of "addition" in the solution above is consistent with the work given within the Rhind Papyrus. The fact that the solution above gives all the unit fraction decompositions in a unified, systematic manner and that there is, at last, a pattern that gives the unit fraction decomposition for $\frac{2}{95}$, as well as others, gives further credibility to the solution process. Lastly, the fact that the initial "addition" method above ceased to work for $\frac{2}{101}$, that the method used for $\frac{2}{101}$ was a straightforward adaptation of the earlier process, and that the method used for $\frac{2}{101}$ could be applied to all fractions $\frac{2}{n}$, where n is odd and greater than 101, could explain why the table ended with the unit fraction decomposition for $\frac{2}{101}$. Whether or not the solution above was used to construct the Rhind Papyrus unit fraction decompositions, the mystery of more than 3656 years now has a solution.

References

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