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# DISK VERSUS FRUSTUM

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Abstract. This note provides a friendly exposition which fills a gap in typical, current calculus texts. We ask and answer the question: "Why, in dealing with a solid of revolution, do we use disks for volume, but frusta for surface area?" We provide a rationale with explicit examples.

## 1. Introduction

At some point in the calculus sequence, we tackle the volume and surface area of a solid of revolution of a plane curve. The infinitesimal element to be integrated is a disk in the case of finding volume, and it is the frustum of a cone in the case of finding surface area. But why do we use two distinct shapes? Until the completion of the first draft of this note, we had not been able to find a standard calculus text which explains this matter (for examples, [1], [4], [5], and [8]). Since then, we became aware that this topic was discussed by George B. Thomas on pp.180-181 of his 1983 text (see [6]). However, it is noteworthy that this topic is omitted or stricken out of later editions or variations of the 1983 text (see [7]). Thus, it seems timely to reprise this matter in the present note, which contains explicit examples and more details than provided in [6].

For simplicity, we assume throughout this article that $f(x) \geq 0$ and $f'(x)$ is continuous for all $x$ on a closed interval $[a, b]$. Let us recall the formulas for the volume $\mathcal{V}$ and the surface area $\mathcal{A}$ of the solid obtained by revolving the curve $y = f(x)$, $a \leq x \leq b$, about the $x-$axis:

$$\mathcal{V} = \int_a^b \pi y^2 \ dx,$$

$$\mathcal{A} = \int_a^b 2\pi y \ dl, \text{ where } dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx.$$

As is well known, $\pi y^2 dx$ represents the volume of an infinitesimal disk, and $2\pi y dl$ represents the surface area of an infinitesimal frustum. See Figure 1 and Figure 2. Intuitively, the frustum is a better approximation than the cylindrical disk both in finding the volume and the surface area, since the frustum is to the disk as the trapezoid is to the rectangle –for calculations of areas under curves. So, why does the cylindrical disk suffice as the elemental solid in finding volume? Dually, one

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can ask why the cylindrical band of a disk is not good enough for the purpose of surface area computation.

## 2. Volume

We first show that disk and frustum methods yield the same result in the calculations of volumes. Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, $x_0 = a$, and $x_n = b$.

The volume $\Delta\mathcal{V}_i$ of the disk (see Figure 1) is

$$\Delta\mathcal{V}_i = \pi(y_i^*)^2\Delta x_i, \text{ where } y_i^* = f(x_i^*), x_i^* \in [x_{i-1}, x_i].$$

Thus, the volume $\mathcal{V}$ using the disks is the following:

$$\mathcal{V} = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta\mathcal{V}_i = \lim_{n\to\infty}\sum_{i=1}^{n}\pi(y_i^*)^2\Delta x_i = \int_a^b \pi y^2 dx.$$

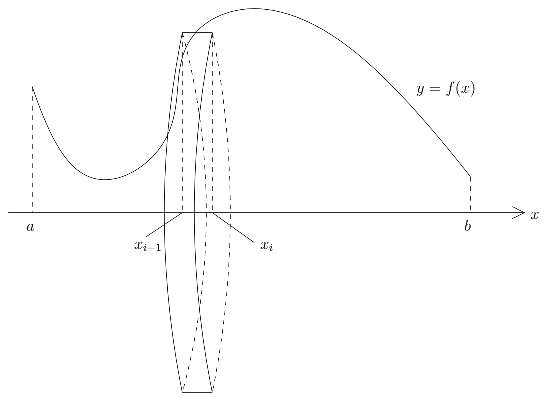


Figure 1

On the other hand, to find the volume of the frustum of a cone (see Figure 2), we first find the $x-$intercept of the line that passes through $(x_{i-1}, y_{i-1})$ and $(x_i, y_i)$. We can write the equation of the line as $y - y_i = \frac{\Delta y_i}{\Delta x_i}(x - x_i)$, and it follows that the $x-$intercept is $x_i - y_i\frac{\Delta x_i}{\Delta y_i}$. Without loss of generality, we may assume $y_i \geq y_{i-1} > 0$. The volume $\Delta\bar{\mathcal{V}}_i$ of the frustum of a cone, obtained by subtracting the volume of the little cone from the volume of the big cone, is

$$\begin{aligned}
\Delta\bar{\mathcal{V}}_i &= \frac{1}{3}\pi y_i^2(y_i\frac{\Delta x_i}{\Delta y_i}) - \frac{1}{3}\pi y_{i-1}^2(x_{i-1} - x_i + y_i\frac{\Delta x_i}{\Delta y_i}) \\
&= \frac{1}{3}\pi y_i^2(y_i\frac{\Delta x_i}{\Delta y_i}) - \frac{1}{3}\pi y_{i-1}^2(-\Delta x_i + y_i\frac{\Delta x_i}{\Delta y_i}) \\
&= \frac{1}{3}\pi\Delta x_i(y_i^3\frac{1}{\Delta y_i} + y_{i-1}^2 - y_{i-1}^2 y_i\frac{1}{\Delta y_i}) \\
&= \frac{1}{3}\pi\Delta x_i(\frac{y_i}{\Delta y_i}(y_i^2 - y_{i-1}^2) + y_{i-1}^2) \\
&= \frac{1}{3}\pi\Delta x_i(y_i^2 + y_{i-1}y_i + y_{i-1}^2).
\end{aligned}$$

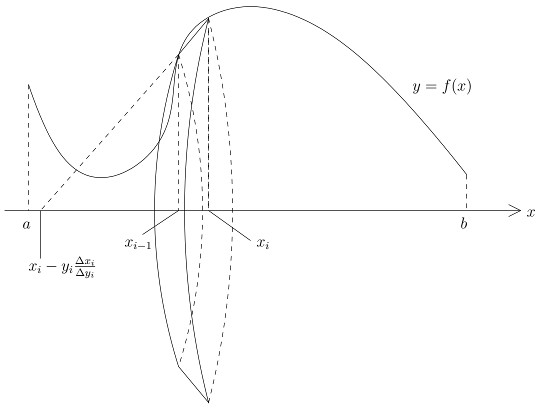


FIGURE 2

Thus, the volume $\bar{\mathcal{V}}$ using the frusta is

$$\begin{aligned} \bar{\mathcal{V}} &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta\bar{\mathcal{V}}_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{3}\pi(y_i^2 + y_{i-1}y_i + y_{i-1}^2)\Delta x_i. \end{aligned} \tag{2.1}$$

Though (2.1) is not a Riemann sum, Bliss' theorem (see [2]) says that it still equals $\int_a^b \pi y^2 dx$. Thus, the disk method and the frustum method yield the same volume.

## 3. Surface Area

We begin by observing that disk and frustum methods do not yield the same result in computing surface areas. The surface area $\Delta\mathcal{A}_i$ of the frustum of a cone (see Figures 2, 3) is

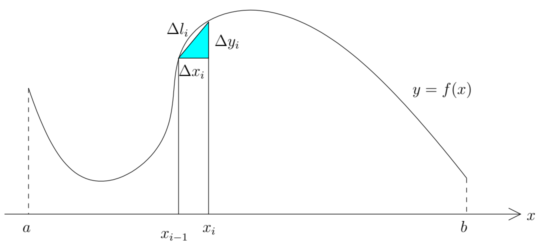


FIGURE 3

$$\Delta\mathcal{A}_i = 2\pi r_i \Delta l_i = 2\pi r_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}, \text{ where } r_i = \frac{f(x_{i-1}) + f(x_i)}{2}. \tag{3.1}$$

Thus, the surface area $\mathcal{A}$ using the frusta is the following:

$$\begin{aligned} \mathcal{A} &= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta\mathcal{A}_i = \lim_{n\to\infty}\sum_{i=1}^{n}\pi(f(x_{i-1})+f(x_i))\sqrt{1+(\frac{\Delta y_i}{\Delta x_i})^2}\ \Delta x_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\pi(f(x_{i-1})+f(x_i))\sqrt{1+[f'(c_i)]^2}\ \Delta x_i \qquad (3.2) \end{aligned}$$

for some $c_i \in (x_{i-1}, x_i)$ by the Mean Value Theorem as applied to $f'$. By Bliss' theorem, (3.2) becomes

$$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\ dx = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx, \qquad (3.3)$$

as $\Delta x_i$ approaches zero.

On the other hand, the surface area $\Delta\bar{\mathcal{A}}_i$ of the cylindrical band of a disk (see Figure 1) is

$$\Delta\bar{\mathcal{A}}_i = 2\pi\bar{r_i}\Delta x_i, \text{ where } \bar{r_i} = f(x_i^*), x_i^* \in [x_{i-1}, x_i]. \qquad (3.4)$$

Thus, the surface area $\bar{\mathcal{A}}$ using the disks is

$$\bar{\mathcal{A}} = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta\bar{\mathcal{A}}_i = \lim_{n\to\infty}\sum_{i=1}^{n}2\pi f(x_i^*)\Delta x_i = \int_a^b 2\pi f(x)dx. \qquad (3.5)$$

Clearly $r_i$ of (3.1) and $\bar{r_i}$ of (3.4) become equal as $\Delta x_i$ approaches zero. However, $\sqrt{(\Delta x_i)^2+(\Delta y_i)^2} - \Delta x_i$ tends to $\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-1\right) dx$ as $\Delta x_i$ approaches zero, and the latter is of course nonzero wherever the curve has a non-horizontal tangent.

As a first example, consider the cone given by revolving about the $x$-axis the line $y = f(x) = mx, m \neq 0$, $0 \leq x \leq b$. The surface area $\mathcal{A}$ using (3.3), the correct formula, gives $\mathcal{A} = \int_0^b 2\pi mx\sqrt{1+m^2}\,dx = \pi m\sqrt{1+m^2}\,b^2$. However, $\bar{\mathcal{A}} = \int_0^b 2\pi mx dx = \pi m b^2$ using (3.5), which is clearly different from $\mathcal{A}$ for $m \neq 0$.

## 4. Further Rationale

To better understand the issue at hand, we consider a viewpoint which presumably dates back to Newton and Leibniz; that is, only differentials of the lowest order need to be kept. Consider $A = xy$ and $A_* = (x+\Delta x)(y+\Delta y)$, we note that $\Delta A = A_* - A = x\Delta y + y\Delta x + \Delta x\Delta y$. This suggests, a priori, that $dA = xdy + ydx + dxdy$. But, of course, $dxdy$ does not actually appear in the product (Leibniz) rule. By the same token, the first order difference in surface area between the elemental frustum and the elemental disk is important; but the difference in volume between the elemental frustum and the elemental disk, being of second order, may be ignored. Towards making clear this notion, we furnish two pieces of computation. First, we show that the frustum integral (3.3) follows from the more formal definition of the area of a parametrized surface. Then, we show a less trivial example (as compared to the cone example) where, of the same surface, the difference between the area as computed via the frustum method and the area as computed via the disk method is seen to be of first order.

Recall the definition of the area of a parametrized surface (see p. 910 of [4]):

**Definition 4.1.** If a smooth parametric surface $S$ is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},\ (u,v) \in D$$

and $S$ is covered just once as $(u,v)$ ranges throughout the parameter domain $D$, then the surface area of $S$ is

$$\mathcal{A}(S) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v|\ dudv, \tag{4.1}$$

where $\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$ and $\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$.

Note that the area so defined is independent of parametrization (see section 7.6 of [3]).

Consider the surface obtained by revolving $y = f(x)$ about the $x-$axis. Then the top half of the surface ($y \geq 0$ portion) is parametrized by $\mathbf{r} : (u,v) \longmapsto (x = u, y = \sqrt{(f(u))^2 - v^2}, z = v)$ over the (generally non-rectangular) domain defined by $u = a, u = b, v = -f(u)$, and $v = f(u)$. Using the above definition, we will show that

$$\int_a^b \int_{-f(u)}^{f(u)} |\mathbf{r}_u \times \mathbf{r}_v|\ dvdu = \pi \int_a^b f(u)\sqrt{1 + (f'(u))^2}\ du. \tag{4.2}$$

Note that $\mathbf{r}_u \times \mathbf{r}_v = \frac{f(u)f'(u)}{\sqrt{(f(u))^2 - v^2}}\mathbf{i} - \mathbf{j} - \frac{v}{\sqrt{(f(u))^2 - v^2}}\mathbf{k}$, and thus

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\frac{(f(u)f'(u))^2}{(f(u))^2 - v^2} + 1 + \frac{v^2}{(f(u))^2 - v^2}} = \sqrt{\frac{(f(u))^2[1 + (f'(u))^2]}{(f(u))^2 - v^2}}.$$

Therefore,

$$\begin{aligned}
\int_{-f(u)}^{f(u)} |\mathbf{r}_u \times \mathbf{r}_v|\ dv &= \int_{-f(u)}^{f(u)} \sqrt{\frac{(f(u))^2[1 + (f'(u))^2]}{(f(u))^2 - v^2}}\ dv,\ \text{put } v = f(u)\sin\theta, \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{(f(u))^2[1 + (f'(u))^2]}{(f(u))^2 - (f(u))^2\sin^2\theta}}\ f(u)\cos\theta\ d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{(f(u))^2[1 + (f'(u))^2]}{(f(u))^2\cos^2\theta}}\ f(u)\cos\theta d\theta \\
&= f(u)\ \sqrt{1 + (f'(u))^2}\ \theta\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \pi f(u)\sqrt{1 + (f'(u))^2},
\end{aligned}$$

and thus (4.2) follows. Remembering that we computed only the area of one of two equal halves of the surface of revolution, the frustum integral (3.3) is thus verified.

Note that we can satisfy a more restrictive definition of a parametrized surface found in some calculus texts (cf. p.1108 of [7]) by pre-composing $\mathbf{r}$ with a change of coordinates map given by $T : (u,v) \longmapsto (u, f(u)(2v-1))$ from $[a,b] \times [0,1]$ to the non-rectangular parametrizing domain.

Now, consider the surface $S$ (a thin horizontal slice of the unit sphere at $\phi = \frac{\pi}{4}$, see Figure 4) defined by $\frac{\pi}{4} \le \phi \le \frac{\pi}{4} + \delta$ and $0 \le \theta \le 2\pi$ on the unit sphere given by $\rho = 1$, where $(\rho, \theta, \phi)$ are the spherical coordinates. We denote by $\mathcal{A}, \mathcal{A}^*$, and $\mathcal{A}^{**}$ the actual area (as defined in (4.1)), the area as given by the elemental frustum, and the area as given by the elemental disk, respectively, for the surface $S$. We will show that

$$\lim_{\delta \to 0} \frac{\mathcal{A} - \mathcal{A}^*}{\delta} = 0 \text{ and } \lim_{\delta \to 0} \frac{\mathcal{A} - \mathcal{A}^{**}}{\delta} \neq 0.$$

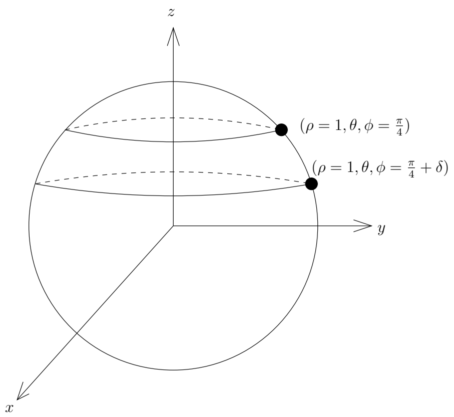


FIGURE 4

To compute $\mathcal{A}$, we use the parametrization $\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k}$.

$$\begin{aligned}\mathcal{A} &= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}+\delta} \sin\phi \; d\phi \; d\theta = \int_0^{2\pi} [-\cos\phi]_{\frac{\pi}{4}}^{\frac{\pi}{4}+\delta} d\theta = 2\pi(\cos\frac{\pi}{4} - \cos(\frac{\pi}{4} + \delta)) \\ &= \sqrt{2}\pi(1 - \cos\delta + \sin\delta) = \sqrt{2}\pi(2\sin^2\frac{\delta}{2} + \sin\delta).\end{aligned}$$

Next, we compute $\mathcal{A}^*$ (see Figure 5).

$$\begin{aligned}\mathcal{A}^* &= \pi(r_1 + r_2)l \\ &= \pi(\sin\frac{\pi}{4} + \sin(\frac{\pi}{4} + \delta))\sqrt{(\sin(\frac{\pi}{4} + \delta) - \sin\frac{\pi}{4})^2 + (\cos(\frac{\pi}{4} + \delta) - \cos\frac{\pi}{4})^2} \\ &= \frac{\sqrt{2}}{2}\pi(1 + \cos\delta + \sin\delta)\sqrt{2(1 - \cos\delta)} \\ &= \frac{\sqrt{2}}{2}\pi(1 + \cos\delta + \sin\delta)\sqrt{4\sin^2\frac{\delta}{2}} \\ &= \sqrt{2}\pi(1 + \cos\delta + \sin\delta)\sin\frac{\delta}{2}.\end{aligned}$$

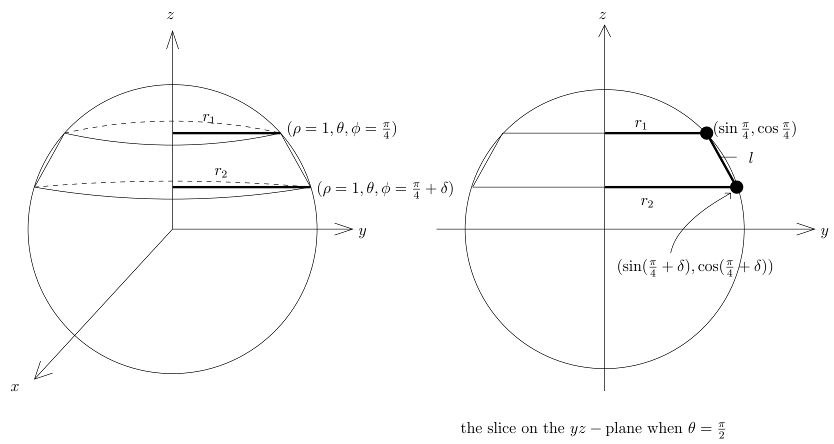


FIGURE 5

Now, we compute $\mathcal{A}^{**}$. For convenience, we take a disk of radius $\frac{1}{2}(r_1 + r_2)$ (see Figure 6). Then

$$\begin{aligned}\mathcal{A}^{**} &= 2\pi\left(\frac{\sin\frac{\pi}{4} + \sin(\frac{\pi}{4}+\delta)}{2}\right)\left(\cos\frac{\pi}{4} - \cos(\frac{\pi}{4}+\delta)\right) \\ &= \frac{\pi}{2}(1+\cos\delta+\sin\delta)(1-\cos\delta+\sin\delta) = \frac{\pi}{2}((1+\sin\delta)^2 - \cos^2\delta) \\ &= \pi(\sin\delta + \sin^2\delta).\end{aligned}$$

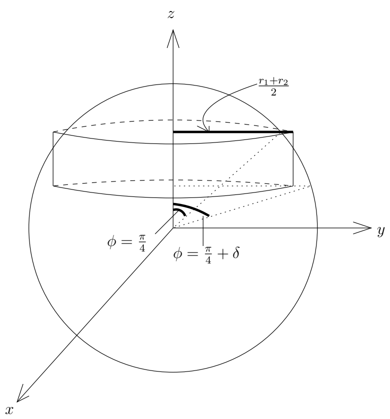


FIGURE 6

Therefore, using $\lim_{\delta\to 0}\frac{\sin\delta}{\delta} = 1$, we have

$$\begin{aligned}\lim_{\delta\to 0}\frac{\mathcal{A}-\mathcal{A}^*}{\delta} &= \lim_{\delta\to 0}\frac{\sqrt{2}\pi(2\sin^2\frac{\delta}{2} + \sin\delta) - \sqrt{2}\pi(1+\cos\delta+\sin\delta)\sin\frac{\delta}{2}}{\delta} \\ &= \sqrt{2}\pi[\lim_{\delta\to 0}(\sin\frac{\delta}{2}) + 1 - \frac{1}{2} - \frac{1}{2}\lim_{\delta\to 0}(\cos\delta) - \frac{1}{2}\lim_{\delta\to 0}(\sin\delta)] \\ &= \sqrt{2}\pi(0 + 1 - \frac{1}{2} - \frac{1}{2} - 0) = 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{\delta\to 0}\frac{\mathcal{A}-\mathcal{A}^{**}}{\delta} &= \lim_{\delta\to 0}\frac{\sqrt{2}\pi(2\sin^2\frac{\delta}{2}+\sin\delta)-\pi(\sin\delta+\sin^2\delta)}{\delta}\\ &= \pi[\sqrt{2}\lim_{\delta\to 0}(\sin\frac{\delta}{2})+\sqrt{2}-1-\lim_{\delta\to 0}(\sin\delta)]\\ &= \pi(\sqrt{2}-1)\neq 0.\end{aligned}$$

It follows that $\lim_{\delta\to 0}\frac{\mathcal{A}^*-\mathcal{A}^{**}}{\delta}\neq 0$, and this is in contrast to the fact that

$$\lim_{\Delta x_i\to 0}\frac{\Delta\mathcal{V}_i-\Delta\bar{\mathcal{V}}_i}{\Delta x_i}=\lim_{\Delta x_i\to 0}\frac{\pi}{3}[((y_i^*)^2-y_i^2)+((y_i^*)^2-y_{i-1}y_i)+((y_i^*)^2-y_{i-1}^2)]=0,$$

where $\Delta\mathcal{V}_i$ and $\Delta\bar{\mathcal{V}}_i$ are as in section 2.

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