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ADVANCED MATHEMATICS RESEARCH EXPERIENCES FOR HIGH SCHOOL STUDENTS

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ABSTRACT: The National Council of Teachers of Mathematics' [NCTM] *Principles and Standards for School Mathematics* (2000) states that the Equity Principle means making challenging mathematics accessible to all students. This includes fostering students' outstanding talent and/or interests in higher mathematics. The purpose of this article is to discuss a method for providing high school students an arena for the study of advanced mathematics through summer mathematics research projects [SMRP].

1. NEED

Prior to more formalized gifted and talented programs, magnet schools, and content academies, students with special talents and/or interests were typically given extra work. That is, if a mathematics student finished his/her work early or mastered the concept as evidenced by homework and tests, then he/she was not necessarily given more challenging extensions or allowed to move forward but instead was simply given more of the same type of assignment. This was a simple solution – keep the student busy. Mathematics education reform efforts support a different answer: nurture the talent and interest through enrichment activities that expand both breadth and depth thus allowing students an opportunity to excel in mathematics [9].

Unfortunately, this is easier said than done. There are several obstacles for high school mathematics teachers. For instance, it is difficult to find the time needed to prepare lessons and activities. It is a well-known fact that teachers are pressed for time and these types of activities require a significant amount of time to research, develop, implement, and evaluate. Moreover, students find themselves in the same position. Students need considerable amount of time to complete in-depth projects and to process ideas. Another impediment may be a void in background knowledge. For example, a student may have a strong inclination to study Combinatorics. It may be difficult for a teacher to develop a comprehensive activity on this topic when teachers' undergraduate preparation usually includes only one course in discrete mathematics.

There is evidence that these barriers are prohibiting the study of higher mathematics or at least that mathematics, as a discipline, is not undergoing the same type of encouragement, as are other science disciplines. Consider University Interscholastic League (http://www.uil.utexas.edu/) competitions such as Number Sense and Calculator. This is an opportunity for students who do well in and enjoy these kinds of mathematics activities to compete with other students. However, these competitions do not necessarily focus on advanced mathematics. Nevertheless, there are formal contests that encourage advanced mathematical topics like the one sponsored by the Siemens Foundation (http://www.siemens-foundation.org/). Yet, if previous years' competitions reflect the amount of higher mathematics that is being studied across the nation, it is a grim picture. In the Siemens Westinghouse Competition in Math, Science, and Technology, there are six regional competitions that lead to one national competition. In 2001 and 2002, the Southwestern Region, which consists of Texas, California, and Oklahoma, had only one final submission in the category of mathematics and in 2003, there were two mathematics submissions.

2. A PROPOSED SOLUTION

The situation is such that there is a need to provide all high school students with an opportunity to pursue advanced studies in mathematics. In order to alleviate the obstacles that high school teachers may face and to increase the number of underrepresented/underserved high school students that study mathematics at a higher level, a possible solution is to have faculty from local colleges/universities provide enrichment activities during the summer.

An activity that proved to be successful is engaging high school students in summer mathematics research projects [SMRP]. In other words, give high school students an opportunity to learn mathematical content that stretches beyond traditional high school mathematics curriculum in a manner that empowers them to access more advanced mathematics. The method includes guidance by university faculty through mathematical experiences that yield personal discoveries resulting in a more thorough understanding. As an example of an effective SMRP, the following is a chronological description of a six-week project that three students participated in during the summer of 2001 at Texas State University-San Marcos. Note that all three students, Cynthia Chi, Charles Hallford, and Rebecca Williams won 1st place in the regional Siemens Westinghouse Competition in Math, Science, and Technology and 4th place nationally. Together with their advisor, they have also published their results in a prestigious research journal [2].

There are both benefits and incentives for all parties involved. For instance, not only did the students receive recognition, they received a monetary award as well. It is also possible to arrange for students to receive dual credit. Teachers that may want to have a similar experience can perhaps gain professional development credit and/or graduate credit. Maximizing the results of the experience is important.

3. SMRP EXAMPLE – THE EDGE SUMS OF DEBRUIJN GRAPHS

Intuitively, a graph is a set of nodes together with a set of links between them that are called edges. Such a general structure provides a solid mathematical model for different applications. For example, interconnection networks are usually modeled by graphs. In particular, many of the graphs used as interconnection network models are defined algebraically in such a way that each node has an integer value. As a result, every edge can be assigned the sum of the two nodes it joins. These numbers are called the edge sums of the graph. The edge sums problem of a graph is to characterize its set of edge sums. Graham and Harary [4] introduced this problem and presented the solution for hypercubes. The objective of the project is to characterize the edge sums for another well-known interconnection network model, the so-called deBruijn graphs.

More formally, a graph G is a pair (V,E) where V is a nonempty set of nodes and $E \subseteq \{\{i,j\}: i,j \in V\}$ is the edge set. Given a numerical labeling of the nodes of a graph G, the network N(G) is constructed by assigning an integer weight to the edges of G as follows. The weight w_{ij} or edge sum of edge $\{i,j\} \in E$ is defined by $w_{ij}=i+j$. With each graph G we can associate a set L(G) of the weights on the edges of N(G). Thus, the edge sums problem consists of the characterization of the set L(G) of all integers that are the weights on the edges of N(G).

The deBruijn graphs can be defined in the following way. Given positive integers d and n where d≥2 and n≥d, the deBruijn graph B(d,n) has for nodes the set of sequences of length n on $Z_{d,}$ the integers modulo d. Two nodes are joined by an edge whenever the last n-1 terms of one of them are identical with the first n-1 terms of the other [3].

From the definition of deBruijn graphs, it follows that the nodes of the deBruijn graph B(d,n) are all the integers between 0 and d^n-1 represented in base d. Then, an edge connects a node i with a node $j_k = ((id) \bmod d^n) + k$, where $0 \le k \le d-1$. Note that multiplying by d means to shift the corresponding sequence one unit to the left, so when adding k, $0 \le k \le d-1$ we obtain all the possible sequences whose first n-1 terms coincide with the last n-1 terms of the node i. With this labeling on the nodes of B(d,n) we consider the network N(B(d,n)) in which the weight of the edge $\{i,j_k\}$ is $w(ij_k)= i + ((id) \bmod d^n) + k$.

As an example, see below the graph B(2,3) and its edge sums.

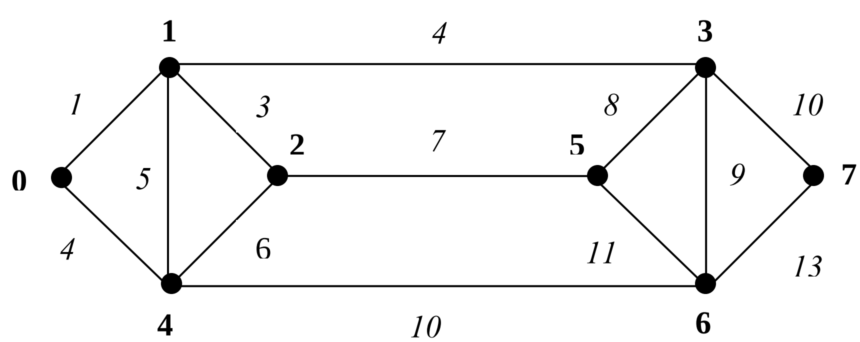

Figure 1 The graph B(2,3) and its edge sums.

Week 1 – Tutorial

The first week consisted of a quick but thorough tutorial in Graph Theory. To alleviate any anxiety the students may have had, the tutorial was carefully designed in order to cover all the concepts needed to understand the research problem, but nothing else. Therefore, no textbook was followed, and the students were provided with a handout that summarized terminology, definitions, and concepts as noted above. In addition, the students were given many challenging exercises to assimilate the ideas and were provided an opportunity to discuss applications of the problem to understand the real-world relevance as well as to increase motivation. For instance, the students explored the mathematics utilized in urban traffic studies and data flow of networks.

Week 2 – Modeling

During the second week, the students were committed to the study of the known results in relation to the research project. Specifically, the students critically read and discussed the work of Graham and Harary [4]. Reviewing previous work served to help the students conduct an integral part of research, the review of literature. Moreover, the process allowed students to learn about how results are achieved and communicated. The review also helped to determine methodology and spark additional questions for further research. By the end of the week, the students were asked to start working on the edge sums problems for the deBruijn graphs B(2,n). This particular case was chosen because it involves working with binary sequences rather than with more complex modular arithmetic. Below is a description of the students' results.

After analyzing several particular examples, the students realized that the best approach to the case of binary deBruijn graphs B(2,n) was to consider two cases depending on n even or odd. The two theorems that follow are the characterization of the edge sums sets.

Theorem 3.1*:* *Let n be an even positive integer, and let L_n be the edge sums set of B(2,n). Then,*

$$L_n = \{m : m \equiv 0 \bmod 3 \text{ or } m \equiv 1 \bmod 3,\ 1 \le m \le 3(2^{n-1}-1)+1\}$$
$$\cup \{m : m \equiv 2 \bmod 3 \text{ or } m \equiv 0 \bmod 3,\ 2^{n-1} \le m \le 3(2^{n-1}-1)+2^{n-1}\}.$$

Theorem 3.2: *Let n be an odd positive integer, and let L_n be the edge sums set of B(2,n). Then,*

$L_n = \{m: m \equiv 0 \bmod 3 \text{ or } m \equiv 1 \bmod 3,\ 1 \leq m \leq 3(2^{n-1}-1)+1\}$

$\cup \{m: m \equiv 1 \bmod 3 \text{ or } m \equiv 2 \bmod 3,\ 2^{n-1} \leq m \leq 3(2^{n-1}-1)+2^{n-1}\}.$

Week 3-6 – Research

The majority of the research was done during the following four weeks. The students were directed to collectively investigate the edge sums problem for B(d,n) using the solutions obtained for the edge sums problem B(2,n). The students discussed their research with the instructor for at least two hours a day. Each of these sessions began with a 15-30 minute status and work report. The report was followed by an in-depth discussion centered on their conjectures. The last 15-30 minutes were spent on maintaining a log of their progress. Throughout the entire meeting the instructor helped the students learn how to clearly present results and ideas with mathematical rigor. Below is a description of the students' results.

Theorem 3.3: *Let d and n be two positive integers. If n is even and $d \geq 2$, then the edge sums set of B(d,n) contains all the integers between 1 and $2d^n - 3$, except for the $2(d^{n-1} - d)/(d+1)$ integers in the set* $\{m: m \equiv 3d \bmod (d+1),\ 0 \leq m \leq d^{n-1} - d\} \cup$

$\{2d^n - 2 - m: m \equiv 3d \bmod (d+1),\ 0 \leq m \leq d^{n-1} - d\}.$

Theorem 3.4: *Let d and n be two positive integers. If n is odd and $d \geq 2$, then the edge sums set of B(d,n) contains all the integers between 1 and $2d^n - 3$, except the $2(d^{n-1} - 1)/(d+1)$ integers in the set* $\{m: m \equiv 3d \bmod (d+1),\ 0 \leq m \leq d^{n-1}-1\} \cup \{2d^n - 2 - m: m \equiv 3d \bmod (d+1),\ 0 \leq m \leq d^{n-1} - 1\}$.

Moreover, the students became interested in a problem, the multiplicity of the edge sums, which initially was not part of the project. The students decided to pose the problem and solve it as well; they were successful in the case of binary deBruijn graphs. The results follow.

Theorem 3.5: *Let n be an even positive integer. The multiplicity of all the integers in the edge sums set of* $B(2,n)$ *is 1, except for the integers in the set* $\{m: m \equiv 0 \bmod 3,\ 1+2^{n-1} \leq m \leq 3(2^{n-1}-1)\}$ *that have multiplicity 2.*

Theorem 3.6: *Let n be an odd positive integer. The multiplicity of all the integers in the edge sums set of* $B(2,n)$ *is 1, except for the integers in the set* $\{m: m \equiv 0 \bmod 3,\ 2^{n-1} \leq m \leq 3(2^{n-1}-1)+1\}$ *that have multiplicity 2.*

4. Suggestions for creating SMRPs

The keys to an effective SMRP are to (1) carefully choose a topic and (2) strategically outline the instructional methodology. The teaching technique is influenced by the topic although there are some commonalities to SMRPs that should be taken into consideration as detailed below.

Initially, a decision must be made that determines what area of mathematics is to be investigated. Taking into consideration that high-school students may not have the mathematical background and/or maturity to delve into theoretical mathematics, the recommendation is to focus on applied mathematics. Moreover, it is important to choose a topic of applied mathematics that lends itself to various applications. Fortunately, most of applied mathematics fields such as Graph Theory and Combinatorics have an assortment of real-world, current applications especially because of their connections to computer science and our technological society.

In terms of teaching technique, three guidelines should be observed. First, the SMRP should be a problem presented at the outset in terms of its applications. It is particularly important to provide students with a context for their problem. Every effort should be made to include every day examples. For instance, the origin of networks is computer-based but it is helpful to discuss its function in phone systems, traffic control, and urban planning. Second, the minimum mathematical requisites to understand the problem should be offered. In addition, background knowledge should be approached intuitively and in a natural setting. In the case of deBruijn networks, one of two approaches can be taken to bring students to a level of understanding that makes network problems attainable. One method involves modular algebra and the other technique deals with representation of integers in different bases. Here, one would choose the latter because topics such as divisibility and multiples related to bases are typically a part of K-12 curriculum standards (Texas Essential Knowledge and Skills [TEKS], http://www.tea.state.tx.us/). The TEKS, which are both content and process standards, include strands such as quantitative and algebraic reasoning. These strands are the foundation for Algebra I & II. Students that receive an “A” in Algebra I & II will be adequately prepared to participate in Discrete Math projects. Furthermore, discussion about bases can include pertinent references to familiar examples such as place value, number systems, and computers. This allows for connections back to the general mathematics topic and for fluid conversation that avoids complicated mathematical terms and notation without the loss of rigor. Third, proven results on related problems should be given. However, this does not include solutions to specific cases since this may stifle creativity and innovation. In NCTM’s *Principles and Standards for School Mathematics* [9], recommendations are made to move students from a concrete experience to a more abstract understanding. In the spirit of this recommendation, students should be encouraged to investigate numerical examples, specific cases, and use inductive reasoning to generalize. Examples of this type of process including results should be demonstrated with parallel topics. In terms of networks, a problem about deBruijn graphs could be accompanied by results for hypercubes.

Doing advanced mathematics with high school students requires special attention to both curriculum and instruction. Coupling research-proven techniques with experience with teaching applied mathematics yields the three recommendations above.

5. Ideas for other SMRPs

Although it is not impossible to create new mathematics as evidenced above, it may be difficult for students to develop new theorems and proofs without a deep understanding of the mathematical foundation. Thus, the majority of the activities that students are engaged in typically centers on mathematics that is new to them but not necessarily new to the field.

To assist teachers and student with project ideas, there are several resources available that focus on mathematics-related competitions and sample problems. A good place to begin a search for these types of materials is the *Mathematics Projects Handbook* by Allinger, et al [1]. This book contains a list of resources including an annotated bibliography of internet sites and journal articles from which to gain a fundamental understanding of what types of mathematics projects exist and how these projects were established, supported, and conducted.

Nevertheless, newer fields such as Graph Theory and Combinatorics provide more opportunities for innovative mathematical endeavors. The following is a list of interesting project ideas in Graph Theory and sources of additional information.

1. Eccentric sequences of connected graphs. Source: Lesniak (1975) [8].
2. Domination number of the n-dimensional hypercube. Source: Harary and Livingston (1993) [6].
3. Connectivity of path graphs. Source: Knor, Niepel and Mallah (2002) [7].

6. SUMMARY

Students that participate in SMRPs are definitely benefiting from the experience. First, the students gain tremendously from the experience of preparing for competition. The students enhance their presentation skills, become more proficient in technical writing, and establish significant working relationships with professionals in the field as well as with their peers. Second, the students are engaged in scholarly activity culminating in a publication. This type of activity contributes to the research knowledge and it is considered in high regard. Third, the students are immersed in a positive learning environment that allows them to ascertain higher level mathematics and increase their self efficacy. By providing high school students access to advanced mathematics research, students are not only learning more mathematics but they are also gaining strategies to approaching higher mathematics especially those that parallel that of professional mathematicians [5]. Thus, students with an SMRP experience are better equipped to succeed in a university setting especially in Mathematics, Science, Engineering, and Technology fields. Having SMRP participants culminate in participating in competition helps to make the entire process a simulation of real-life challenges as well as inspirational and rewarding.

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