

THE BUFFON NEEDLE PROBLEM EXTENDED

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ABSTRACT. Geometrical statistical methods are used to study needles floating in a weightless environment. This is a three dimensional analog of the classical Buffon Needle problem in two dimensions.

1. INTRODUCTION

In the 1770’s G. Buffon of France proposed an experiment to statistically compute the number π . The Buffon Needle Problem rated entry #18 in the 100 Great Problems in Elementary Mathematics by H. Dörrie [1]. Historically this problem was the first use of geometrical methods in statistics and its proof utilized the then newly innovated tool of integral calculus.

Buffon’s Needle Problem can be performed empirically by first drawing parallel lines one unit of length apart on a plane surface and randomly dropping needles of unit length on the surface. Count the number of needles dropped on the surface, multiply this number by 2. Use this number as the numerator of a ratio. Count the number of needles that touch a line. Use this count for the dominator of the ratio. The expected value of this quantity is π .

$$\pi = \frac{2 * \text{all needles}}{\text{touching needles}}$$

The Buffon Needle Problem has been extended to include non-unit line spacing and needles of different lengths [2].

The purpose of this paper is to extend the problem to a weightless environment. This adds a third dimension to the problem because the needles would float. How would the parameters of the experiment change? Would this experiment yield such an interesting ratio?

2. BUFFON NEEDLE PROBLEM

When needles are dropped on a plane, they can fall with random angles relative to the angle of the parallel lines, θ , and random distances from the center of the needle to the closest point on the line, y . The probability that the tip of the needle will touch one of the lines depends on these two random variables. The angle θ that the needle makes with the x-axis, which is parallel to the lines on the plane, can range from 0 to π . After the needle rotates through an angle of π radians, it essentially returns to the original orientation due to the symmetry of the needle.

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The distance of the center of the needle to the closest line ranges from 0 to $\frac{1}{2}$. If y is any greater, it would be closer to the next line. The x-coordinate of the needle does not effect whether it touches one of the parallel lines.

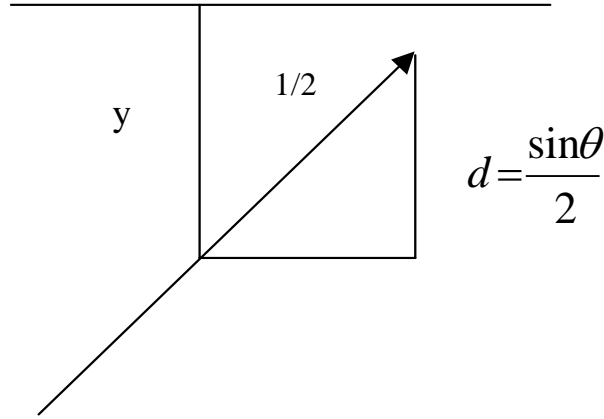
Since θ and y are independent variables and presumed to be uniformly distributed, the measure of the sample space is

$$(\pi) \left(\frac{1}{2} \right) = \frac{\pi}{2}.$$

The conditions that the needle touches a line would require

$$y \leq d = \frac{1}{2} \sin \theta.$$

See Figure 1.



Lines are one unit apart. Unit Length Needle.

Figure 1.

The measure of the space that meet the conditions for a touch is

$$\int_0^{\pi} \int_0^{\frac{1}{2} \sin \theta} dy d\theta = 1.$$

The probability that a needle will touch a line then is

$$\frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

The empirical probability that the needles touches can be equated to this ratio.

$$\frac{\text{touching needles}}{\text{all needles}} = \frac{2}{\pi}$$

Solving for π , we get the formula

$$\pi = \frac{\text{touching needles}}{\text{all needles}}.$$

3. EXTENDING THE PROBLEM TO A WEIGHTLESS ENVIRONMENT

If a needle were released in a weightless environment, then it wouldn't drop down to the plane, it would float. This introduces another dimension into the problem. We add a z-axis to the picture.

What is the three dimensional analog of the needle touching a line in two dimensions? There may be several possible choices. Would it be that the needles touch the surface of a sphere? We choose to transform the parallel lines into planes perpendicular to the original plane intersecting the original plane at the original lines.

Instead of dropping flat on the plane, the needles now have an orientation relative to the z-axis, ϕ . We also assume this variable to be independent of the others and uniformly distributed

$$0 \leq \phi \leq \pi$$

The size of the sample space does not depend on the z-coordinate. So the measure of the sample space is

$$(\pi)(\pi)(1/2) = \frac{\pi^2}{2}$$

The criterion for touching a plane still comes down to the y coordinate of the projection of the center of the needle on the x-y plane being less than a distance 1/2 to the nearest line. The measure of the touches is

$$\int_0^\pi \int_0^\pi \int_0^{\frac{1}{2} \sin \theta \sin \phi} dy d\theta d\phi = 2.$$

The probability of a touch is the ratio of the measure of the touching needles over the measure of the sample space.

$$\frac{2}{\frac{\pi^2}{2}} = \frac{4}{\pi^2} = \left(\frac{2}{\pi}\right)^2$$

This is the extension of the Buffon Needle Problem to the weightless three dimensional case. Notice adding a third dimension added a power to the result of the two dimensional case.

Equating the empirical probability to this ratio gives the following equation

$$\frac{\text{touching needles}}{\text{all needles}} = \left(\frac{2}{\pi}\right)^2.$$

Solve for π ,

$$\pi = \sqrt{\frac{4 * \text{all needles}}{\text{touching needles}}}.$$

4. THE TALK GIVEN AT THE TEXAS SECTION MEETING

The original reason for looking at extending the Buffon Needle Problem to a weightless environment was to design a mathematical experiment in which college level students could participate that could fly on a space flight or in a weightless environment such as NASA's KC-135 Vomit Comet, an airplane in a parabolic trajectory.

How would you count the needles touching the parallel planes? One possibility is to cut deep slits into the floor plane and shine a light from below. The light coming out of the slit would form the vertical planes. The floating needles that are lit would count as touches. The needles could be photographed to freeze the rotations in time so that the touches could be counted later.

This talk was originally given at the Texas Section of the MAA. D Charles Bedard, Texas A&M University - Kingsville suggested to make the light source a laser pulsing from below. Then you could take a series of photographs with the different pulses and thus enlarge the sample size.

Another problem raised was randomizing the angles of the needles in space. Keith E. Emmert, Texas Tech University suggested a possibility of replacing needles with trails left by radio active decay in a cloud chamber. The radio active decay would provide random angles.

At the talk I performed the original Buffon Needle experiment on an overhead projector, with 50 "needles". I also presented a Monte Carlo model of the weightless three dimensional case using the assumptions in the talk and taking 10000 samples using Microsoft Excel. The 10000 samples produced 4010 "touches" for an empirical probability of 0.4010 compared to a theoretical probability of about 0.4053 for $(\frac{2}{\pi})^2$.

The Buffon Needle problem in three dimensions is basically a binomial distribution with the probability of a touch, p , being $(\frac{2}{\pi})^2$ and the probability of no touch is $1 - (\frac{2}{\pi})^2$.

The mean, μ , for this experiment is

$$\mu = np.$$

The standard deviation, σ , is

$$\sigma = \sqrt{np(1-p)}.$$

For large sample sizes, n , and since p is "close" to .5, we can use the formulas for the normal approximation of the binomial distribution [3].

The sample proportion of touches to needles, $\frac{x}{n}$, for our Monte Carlo run was

$$\frac{4010}{10000} = 0.4010.$$

The confidence interval for p would be

$$\frac{x}{n} - z\sqrt{\frac{\frac{x}{n}(1-\frac{x}{n})}{n}} < p < \frac{x}{n} + z\sqrt{\frac{\frac{x}{n}(1-\frac{x}{n})}{n}}$$

where z is in units of standard deviation from the mean. For a 95% confidence interval

$$z = 1.96.$$

The 95% confidence interval for our Monte Carlo experiment would be

$$.4010 - 1.96\sqrt{\frac{.4010(1-.4010)}{10000}} < p < .4010 + 1.96\sqrt{\frac{.4010(1-.4010)}{10000}},$$

$$0.39139 < p < .41061.$$

Using this range of values for a sample size of 10000 in our computing formula for π ,

$$\pi = \sqrt{\frac{4 * \text{all needles}}{\text{touching needles}}}$$

We now have an idea how accurate our estimate for π would be using a three dimensional weightless experiment.

$$\sqrt{\frac{4 * (10000)}{4106.1}} < \pi < \sqrt{\frac{4 * (10000)}{3913.9}}$$

$$3.1212 < \pi < 3.1969$$

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