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SUBTRACTIVE BLACK HOLES AND BLACK LOOPS

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ABSTRACT. Subtractive black holes and black loops are generalizations of Kaprekar's constants. When the subtractive Kaprekar routine is applied to an *n*-digit number a termination in the form of a black hole or black loop is reached. If all *n*-digit numbers for a specific value of *n*, other than numbers where all digits are the same, result in the same black hole then that number is a Kaprekar constant. The process and its results provide a rich environment for conjecture and proof.

For mathematics instructors, finding a topic that not only offers a real mathematical challenge but can also be readily understood by students is an important discovery. This situation is especially true in pre-college teaching, and a good example of this type of topic is the Goldbach Conjecture [6]. Although unproven for over 250 years, this conjecture requires only that students be able to identify prime numbers and add in order to explore its workings. For this reason, this topic is accessible even to students at the elementary school level; yet mathematicians still search for a proof of the conjecture.

The purpose of this article is to introduce "Subtractive Black Holes and Black Loops", an activity that can be understood by third grade students yet extensions of the activity are today being investigated by research mathematicians.

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The activity is started by selecting a three-digit number, where all the digits are not the same. Arrange the digits in descending order so as to provide the greatest three-digit number that can be composed from those three digits. Then reverse the order of the digits to provide the least three-digit number that can be created from these same three digits. Subtract the lesser number from the greater one and repeat this "subtractive process" until an "interesting" result is obtained. For example:

1) Select 492 (which yields 942 as the greatest and 249 as the least possible number from the digits 4, 9 and 2)
2) 942 – 249 = 693
3) 963 – 369 = 594
4) 954 – 459 = 495

Since 495 and 594 are composed of the same three digits, repeating the subtractive process will continue to yield 495. Ultimately, students will discover that any three-digit number (other than numbers where all three digits are the same), when subjected to the subtractive process described above, will yield 495 in, at most, six steps. Thus, the number 495 is termed a three-digit "black hole." The proof that 495 is the black hole for all three-digit numbers, with all three digits not the same, is as follows:

1. Given *abc*, a three-digit number such that $a \geq b \geq c$; $a > c$.
   a. The difference in the one's place would be $(10 + c) – a$, since $c < a$.
   b. The difference in the ten's place would be $(10 + b – 1) – b = 9$.
   c. The difference in the hundred's place would be $(a – 1) – c$.
2. Step #2b proves that the ten's place digit, under the subtractive process, will always be 9.
3. The results of steps #2a and #2c prove that the sum of the one's and hundred's place digits will always be 9 $[(10 + c) – a + (a – 1) - c = 10 – 1 = 9]$.
4. The only three-digit numbers, with the digits in descending order, of the form *x*9*y*, such that $x + y = 9$ are 990, 891, 792, 693 and 594.
5. Under the subtractive process, 990 yields 891, 891 yields 792, 792 yields 693, 693 yields 594, and 594 yields 495.
6. Since 594 and 495 comprise the same digits, then the termination for the subtractive process must necessarily be 495.

One of the authors of this article first learned of the black hole phenomenon from an NCTM publication in 1993 [5]. An article in this publication demonstrated that the subtractive process applied to four-digit numbers would lead to a black hole of 6174. The four-digit subtractive process was used to reinforce subtraction skills for students in grades four through eight. Instead of providing the usual "boring" subtraction exercises so often employed in mathematics classes, instructors presented the subtractive process to students and then asked them to find a four-digit number for which the process would **NOT** lead to 6174. These students worked many a subtraction exercise during their search; however, not one of the students voiced a single complaint.

In 1995, fourth-grade students at Dunleith Elementary School in Marietta, GA were asked to investigate the subtractive process using three-digit numbers. At that time, their teacher was not aware of whether a similar black hole number would be produced or what that number might be. After exploring several student-generated results, the termination was clear: the process always led to 495. Although these elementary students "discovered" the three-digit black hole, the result was first published in 1974 [7]. There was now a three-digit and a four-digit black hole, and this information was presented to teachers at various workshops and conferences across the country. It should be noted that teachers attending these sessions and workshops had not seen this activity before although the four-digit black hole had been explained in an article in *Scientific American* [2].

At this point, the next question to be answered was: What is the five-digit black hole? In September 2000, students at Sam Houston State University attending a course entitled "Introduction to the Foundations of Mathematics" examined the five-digit subtractive process. These are students who plan to teach elementary school. What they discovered was unforeseen by the instructor. No five-digit black hole appeared to exist. Instead, the students initially found two loops (labeled by one of the authors as "black loops"), each containing four elements that were terminations for the subtractive process. One loop comprised the numbers 75933, 63954, 61974 and 82962, while the other loop consisted of 83952, 74943, 62964 and 71973. One student then found a third loop involving only two elements: 53955 and 59994.

Did these results represent all possible five-digit black loops? Would an as yet undiscovered five-digit black hole be revealed? In order to investigate all five-digit numbers, one of the authors of this article wrote a program to determine the termination for each five-digit number. A pseudocode description follows:

Function sort

[0]<u>Computing Intermediate results</u>

```
    // Comment: compute the difference
    Convert the number to a string of digits strVal
    Sort digits into descending order[2]
    Convert string back to integer val
    Reverse order of strVal
    Convert reversed string to number val2
    Subtract val-val2 to produce result
```

[1]<u>Locating black holes and loops</u>

```
  //Comment:  check if we have the value already
  Convert result into a string strResult
  Sort digits of strResult into descending order[2]
  Test strResult against prior elements in the sequence
  If strResult is already in the list      // Comment we have a loop or hole
    If strResult is in the list of known loops and holes
            Discard strResult

  Else
            Add strResult to known list of loops and holes
    End if
    Finished looking for this black hole or loop
  Else
    Place strResult in list   //Comment:  we have a new value so record it
    Still looking for a black hole or loop
  End if
```

[2]<u>Sorting digits</u>

```
For each position n in the list
  Locate the position m of the largest element between position n and the end of the list
  Swap item at n with the item at m
End for
```

<u>Main program</u>

```
For each x in the interesting range
    While we are still looking for a black hole[1]
        Compute Intermediate Results[0] and test for black hole or loop
    End while
End For
```

Running the program revealed that approximately 45% of the five-digit numbers terminated at loop A (75933-63954-61974-82962), about 50% terminated at loop B (83952-74943-62964-71973) and about 5% terminated at loop C (53955-59994). The five-digit numbers that were input represented the greatest number that could be written using the given digits. All other permutations consisting of the same digits were excluded, since they would have the same result.

Observing the behavior of eight numbers between 77630 and 77643 produced the following results:

Number	Loop	Number	Loop
77630	A	77640	A
77631	A	77641	B
77632	C	77642	A
77633	A	77643	B

This sample demonstrates the apparent randomness of the nature of a number's termination. An examination of the five-digit numbers and their terminations has not yet yielded a clear pattern as to which characteristic of a number determines its termination path.

At this point, the question was posed: What is known about this phenomenon? One of the editors of the 1993 NCTM newsletter that contained the information about four-digit black holes was contacted. The information this individual provided led to a now-defunct mathematics/science calendar, which eventually led to a conversation with Ross Honsberger. Professor Honsberger is the author of *Ingenuity in Mathematics* [3], in which 6174 is shown to be the four-digit black hole. In that same book, he briefly mentions that the subtractive process for six-digit numbers leads to black holes of 631764 and 549945, as well as a seven-element loop.

Although Honsberger indicated that he had no knowledge of any investigations of the nature of these loops, he nevertheless provided a useful hint. He said that an Indian mathematician named D. R. Kaprekar initiated the investigation of this phenomenon [4]. What has been referred to in this article as the "subtractive process" should be more correctly identified as the "Kaprekar routine". This name led to communication with Byron L. Walden [8], who has been researching Kaprekar's constants. These constants represent black holes for *n*-digit numbers in base *b* where all the numbers, under the subtractive process, lead to the **same** black hole. Thus 6174 and 495 are Kaprekar's constants for base 10. The number 631764, however, is not a Kaprekar's constant in base 10 because it does not represent a *unique* termination for all six-digit numbers. A seven-digit Kaprekar's constant in base 4 and a nine-digit Kaprekar's constant in base 5 have been identified.

Subtractive Black Holes and Black Loops thus represent phenomena that are accessible and of real interest to elementary school students and yet hold great potential and fascination for research mathematicians. Three basic questions remain under investigation. First, are there Kaprekar's constants that are not yet discovered? In other words, for a specific natural number, *n*, and a specific base, *b*, is there a unique black hole at which all *n*-digit numbers in base *b* will terminate under the subtractive process? The second question focuses on whether**,** for a specific natural number, *n*, the number of black holes and the nature of those black holes can be determined for an *n*-digit number under the subtractive process. Finally, for a natural number, *n*, for which there are multiple terminations under the subtractive process, can the particular termination for a specific number be predicted?

The "Math World" web site created by Wolfram Research [9] provides insight into some of the questions that can be asked about Kaprekar constants, black holes and black loops. Recently, a novel approach was taken by Deutsch and Goldman in the investigation of the Kaprekar routine [1]. These authors classified numbers using equivalence classes and then examined the "orbits' of the equivalence classes. The orbits are displayed graphically, using color codes. Questions related to the Kaprekar routine will undoubtedly continue to captivate, motivate and inspire mathematics students, instructors and researchers alike.

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