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A MAGNUS EMBEDDING THEOREM FOR SECOND HOMOTOPY MODULES

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Abstract. Let G be any group, and $\mathcal{P}$ a presentation for the group. Every group presentation gives rise to a connected two-dimensional CW-complex, $X_{\mathcal{P}}$, with fundamental group, $\pi_1 X_{\mathcal{P}} \cong G$, in a standard way. These two-dimensional CW-complexes are called $[G, 2]$-complexes. We will restrict our attention to these since all two-dimensional CW-complexes with $\pi_1 X \cong G$ are homotopy equivalent to $[G, 2]$-complexes. An open problem in low-dimensional topology is the classification of the homotopy type of $[G, 2]$-complexes. Some progress has been made on this problem in different contexts. In this paper, we examine the Magnus Embedding Theorem, its application to this problem, and extend it to an embedding of the second homotopy module.

1. Introduction

This paper addresses the following question: given two connected two-dimensional CW-complexes, X and Y, with $\pi_1 X \cong \pi_1 Y \cong G$ and $\chi(X) = \chi(Y)$ is $X \simeq Y$? We begin by recalling some definitions and then provide some examples of known results. We focus on results of Dunwoody, et. al., who were the first to demonstrate an example of homotopy inequivalent two-complexes with non-minimal Euler characteristic. To this end, we recall the Magnus embedding theorem, a method of embedding the relation module for a presentation of a group G into a free right $\mathbb{Z}G-$module. Dunwoody uses this as his major tool to prove homotopy inequivalence.

We recognize the limitations of Dunwoody's method, namely that he relies upon an embedding of the relation module instead of the second homotopy module, and therefore recall the notion of crossed modules, which provide us with a presentation of the second homotopy module.

Finally, we state a new application of the Magnus Embedding Theorem, which provides an embedding of the second homotopy module directly. We then provide a simple example of the method, and mention some areas of future exploration.

2. Preliminaries

As mentioned above, we begin with two connected two-dimensional CW-complexes, X and Y, with $\pi_1 X \cong \pi_1 Y \cong G$ and $\chi(X) = \chi(Y)$, and ask: is $X \simeq Y$? Attempting to classify connected two-dimensional CW-complexes, hereafter called

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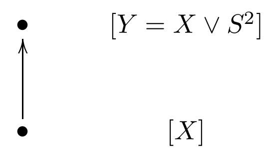
FIGURE 1. Definition of Homotopy Tree



two-complexes, by homotopy type is a daunting prospect, so we note the following well-known theorem:

Theorem 2.1. *Every two-complex is homotopy equivalent to the standard two-complex of a presentation of a group.*

To this end, let G be any group. This will be the fundamental group of our two-complex. Recall that a group presentation for the group G is $\mathcal{P} = \langle \underline{\underline{x}} \mid \underline{\underline{r}} \rangle$ where $r \in \underline{\underline{r}}$ is a word in the $\{\underline{\underline{x}}, \underline{\underline{x}}^{-1}\}$ and $G \cong F(\underline{\underline{x}})/\langle\langle \underline{\underline{r}} \rangle\rangle_F$. We call $\underline{\underline{x}}$ the set of generators and $\underline{\underline{r}}$ the set of relators. The standard two-complex, $X_{\mathcal{P}}$, of a group presentation, $\mathcal{P}$, is built from a single $0-$cell, a $1-$cell for each generator $x \in \underline{\underline{x}}$, and a $2-$cell for each relator $r \in \underline{\underline{r}}$ attached to the $1-$skeleton via its boundary word.

We can begin our classification of two-complexes by first examining the Euler characteristic, which is easy to calculate from the presentation of the fundamental group. For a group G with presentation $\mathcal{P} = \langle \underline{\underline{x}} \mid \underline{\underline{r}} \rangle$, the deficiency of the presentation is $\operatorname{def} \mathcal{P} = \mid \underline{\underline{r}} \mid - \mid \underline{\underline{x}} \mid$. The Euler characteristic is then $\chi(X_{\mathcal{P}}) = \operatorname{def} \mathcal{P} + 1$. The deficiency of a group G is given by

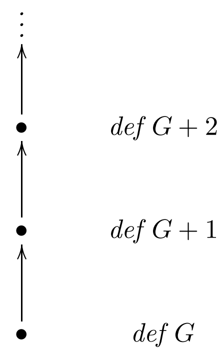
$$\operatorname{def} G = \min \{ \operatorname{def} \mathcal{P} \mid \mathcal{P} \text{ is a presentation of } G\}.$$

It is well known that every group has a minimal presentation, which has minimum deficiency, and hence minimal Euler characteristic. It is easy to see that every group has an infinite number of presentations, each of which can be obtained from a known presentation by inserting the trivial relator into the presentation. In other words, if $\mathcal{P} = \langle \underline{\underline{x}} \mid \underline{\underline{r}} \rangle$ is a presentation for G, then $\mathcal{P}' = \langle \underline{\underline{x}} \mid \underline{\underline{r}}, 1 \rangle$ is also a presentation for G. Moreover, if $X_{\mathcal{P}}$ is the standard two-complex of the presentation $\mathcal{P}$, then $X_{\mathcal{P}'} \simeq X_{\mathcal{P}} \vee S^2$. Furthermore, if $m = \operatorname{def} G$ then for every $n \geq m$ there exists a presentation $\mathcal{P}_n$ of G with $\operatorname{def} \mathcal{P}_n = n$. It is also well-known that if $\mathcal{P}$ and $\mathcal{Q}$ are both presentations of G, then there exists integers r and s so that $X_{\mathcal{P}} \vee \bigvee_r S^2 \simeq X_{\mathcal{Q}} \vee \bigvee_s S^2$.

The homotopy tree of finite $[G, 2]-$complexes, denoted $HT[G, 2]$, is a directed tree with vertices in one-to-one correspondence with the homotopy types of finite $[G, 2]-$complexes, and with a directed edge from the vertex labeled by $[X]$ to the vertex labeled by $[Y]$ whenever $X \vee S^2 \simeq Y$. See Figure 1. It can be shown that the homotopy tree is a connected graph with no cycles, hence is truly a tree.

Since we noted earlier that there exists a minimal Euler characteristic, we can denote the levels of the tree by how far above $\chi_{\min}(G)$ we are. So we set level 0 at $\chi_{\min}(G)$. There is no maximal level on the homotopy tree, since to get to each new level, we can continue to attach an S^2 to a complexes already appearing in the tree, which continues to create new vertices in the tree. We will call a vertex in the tree $HT[G, 2]$ which has no predecessor a root.

FIGURE 2. The Homotopy Tree for a Finite or Free Group



Restating in these new terms, we recall that our motivating question is: If $\mathcal{P}$ and $\mathcal{Q}$ present the group G, and $X_{\mathcal{P}}$ and $X_{\mathcal{Q}}$ have the same Euler characteristic, is $X_{\mathcal{P}} \simeq X_{\mathcal{Q}}$? There are some known results, although the question is still open in many cases. The following results are included to improve exposition, and represent a sampling of previously solved cases.

Theorem 2.2. [2] *For any two non-minimal presentations, $\mathcal{P}$ and $\mathcal{Q}$, of a finite or free group G, if the presentations have the same deficiency then $X_{\mathcal{P}} \simeq X_{\mathcal{Q}}$. The homotopy tree for a finite or free group is shown in Figure 2.*

Theorem 2.3. [12] *The group $G \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ has two homotopy types at the minimal level which arise from the presentations*

$$\langle x, y, z \mid [x, y], [x, z], [y, z], x^5, y^5, z^5\rangle \text{ and } \langle x, y, z \mid [x^2, y], [x, z], [y, z], x^5, y^5, z^5\rangle.$$

Theorem 2.4. [12], [15] *If G is a finite abelian group, then there may be a number of roots at the minimal level, but this number is finite and is characterized by the order of the bias group. Therefore, $HT[G, 2]$ in this case appears is in Figure 3.*

In a series of papers in the 1970's, the first example of homotopy inequivalent two-complexes with non-minimal Euler characteristic was provided.

Theorem 2.5. [1], [5], [6], [7] *For the fundamental group of the trefoil knot complement, there are two homotopy inequivalent two-complexes at level one above minimal Euler characteristic. Therefore, we have $HT[T, 2]$ as seen in figure 4.*

We focus on the results of Dunwoody et. al., which rely heavily on the Magnus Embedding Theorem. They were able to find a second presentation of the fundamental group of the trefoil knot complement (hereafter called the trefoil knot group), T, at level one above the minimal Euler characteristic to compare to the standard presentation at that level, namely the presentation $\langle x, y \mid x^2y^{-3}, 1\rangle$. Since this standard presentation has standard two-complex with second homotopy module requiring only one generator, it remains to show that the standard two-complex of their new presentation requires two or more generators in order to prove that these complexes are homotopy inequivalent. The proof of two generation relies upon the fact that the second homotopy module is isomorphic to the relation module of another presentation of the trefoil knot group, and that this relation module requires

Figure 3. The Homotopy Tree of a Finite Group

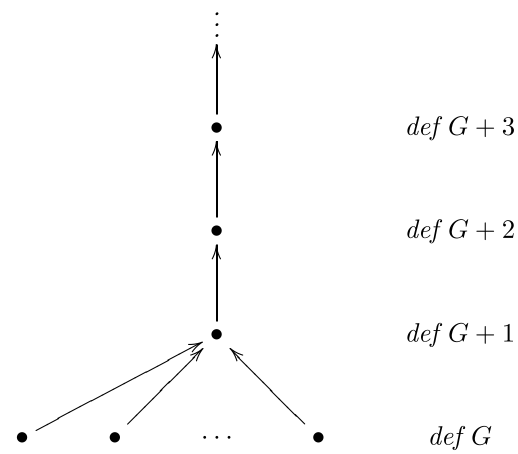
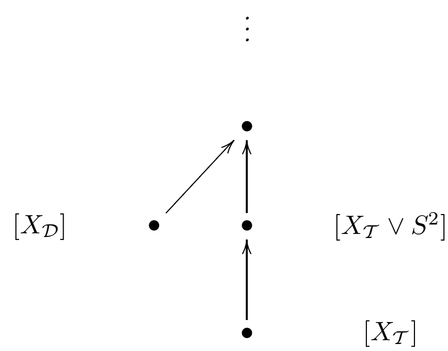


Figure 4. The Homotopy Tree of the Trefoil Knot Group



two generators. This last portion of the proof relies on the Magnus Embedding Theorem to embed the relation module into a free right $\mathbb{Z}T-$module.

In the next section, we will recall the Magnus Embedding Theorem as it was used by Dunwoody. We notice that the application of this Magnus Embedding Theorem to homotopy classification is limited because it requires finding a presentation of the group in which the second homotopy module of the standard two-complex is isomorphic to the relation module of some presentation of the group. The Magnus Embedding Theorem in its original form can then be used to examine the required number of generators of this module. We will extend this embedding theorem to be an embedding of a presentation of the second homotopy module directly, without this intermediate step being required.

3. Magnus Embedding Theorem

The following discussion of the Magnus Embedding Theorem is taken from Magnus' 1939 paper, [10]. To be consistent with Magnus and Dunwoody, we use right actions.

Theorem 3.1. [10] *Let F be a free group on a set of generators $\{x_i\}_{i\in I}$. Let R be a non-trivial normal subgroup of F, $R' = [R,R]$, and $G = F/R$. Then if M is a free right $\mathbb{Z}G$-module generated by $\{t_i\}_{i\in I}$, there is an embedding of F/R' into a group of 2×2-matrices given by:*

$$x_i \mapsto \begin{pmatrix} Rx_i & 0 \\ t_i & 1 \end{pmatrix}$$

i.e. there exists a map

$$(x_i)\varphi \mapsto \begin{pmatrix} Rx_i & 0 \\ t_i & 1 \end{pmatrix}$$

which has kernel $R' = [R,R]$.

Discussion of the properties of the Magnus Embedding can be found in [4] and [10], but we will include those that are important for our calculations later. For our discussion, let F be a free group on n generators, R a non-trivial normal subgroup, $R' = [R,R]$, and $\overline{G} \cong F/R'$. Let φ be the map of F into the matrix group as described above.

Notice that if x is a generator of F, then $(x^{-1})\varphi = \begin{pmatrix} Rx^{-1} & 0 \\ -t_x x^{-1} & 1 \end{pmatrix}$. Also notice if x_i and x_j are elements of $\underline{\underline{x}}$, then $(x_i x_j)\varphi = \begin{pmatrix} Rx_i x_j & 0 \\ t_j + t_i x_j & 1 \end{pmatrix}$. Furthermore,

$$(x^{-1}yx)\varphi = \begin{pmatrix} Rx^{-1}yx & 0 \\ -t_x x^{-1}yx + t_y x + t_x & 1 \end{pmatrix}.$$

Moreover, if w is a relation (i.e. $w \in R$) then $(w)\varphi = \begin{pmatrix} R1 & 0 \\ L & 1 \end{pmatrix}$, where L is a linear function in the t_i given by the right Fox derivative of w.

This last fact is especially important, since Dunwoody is trying to identify the image of elements of the relation module R/R' under the embedding. These elements are first recognized by the 1 in the upper left corner. He then uses the corresponding elements in the free right $\mathbb{Z}G-$module, the entries in the lower left corner of the matrix, to examine the required number of generators of the relation module for his case.

We note that the matrix notation is another way of denoting the semidirect product, $G \rtimes M$. Let M be a free right $\mathbb{Z}G-$module. Then elements of $G \rtimes M$ are elements $(g,m) \in G \times M$ with the operation $(g,m)(g',m') = (gg', mg' + m')$. Notice in the above case we have that $G = F/R$ where F is a free group and R is a nontrivial normal subgroup. The Magnus embedding, φ, is then a map from G into F/R', where $R' = [R,R]$ is the commutator subgroup. This map takes the element $R'x_i$ to the element (Rx_i, t_i) in the direct product of G and M. The semi-direct product multiplication is the same as the matrix multiplication discussed above. We will continue to use the matrix notation, but could use the semi-direct product notation for the remainder of the paper.

To extend the Magnus Embedding Theorem to the second homotopy module instead of the relation module, we need to recall the relationship between crossed modules and the second homotopy module. We will then use crossed modules to apply the Magnus embedding theorem to a presentation of the second homotopy module.

4. Crossed Modules

Again notice that, to later be able to apply the Magnus Embedding Theorem, we use right actions.

Definition 4.1. A crossed module is a triple (G, δ, C) where C and G are groups, with a right action of G on C, $c \cdot g$, and a group homomorphism $\delta : C \to G$ so that the following two conditions are satisfied:

(CM1) $(c \cdot g)\delta = g^{-1}(c)\delta g$ for all $g \in G$ and all $c \in C$
(CM2) $c^{-1}dc = d \cdot (c)\delta$ for all $c, d \in C$

A crossed module is free if it has a basis satisfying the canonical property. Examples of free crossed modules include free modules over a group, free groups, and knot groups. We will focus on the free crossed module $\left(\pi_2(X_{\mathcal{Q}}, X_{\mathcal{Q}}^{(1)}), \delta, \pi_1 X_{\mathcal{Q}}^{(1)}\right)$ which is isomorphic to the crossed module constructed from a presentation $\mathcal{Q}$ as follows.

Define $E(\mathcal{Q}) = F(F(\underline{\underline{x}}) \times \underline{r})$ to be the free group on the set of ordered pairs, (w, r) with $w \in F(\underline{\underline{x}})$ and $r \in \underline{r}$. Let the $F(\underline{\underline{x}})-$homomorphism $\delta : E(\mathcal{Q}) \to F(\underline{\underline{x}})$ be defined on generators by $(w, r) \mapsto w^{-1}rw$. Then $I(\mathcal{Q}) = \ker\, \delta$ is called the group of identities. Let F act on $E(\mathcal{Q})$ by $(w, r) \cdot v = (wv, r)$. Let $C(\mathcal{Q}) = E(\mathcal{Q})/P(\mathcal{Q})$ where $P(\mathcal{Q}) = \langle\langle (V \cdot (U)\delta)^{-1} U^{-1} V U \rangle\rangle_{E(\mathcal{Q})}$ for $U, V \in E(\mathcal{Q})$. The elements of $P(\mathcal{Q})$ are called the Peiffer identities. Whitehead, [16], showed that $(C(\mathcal{Q}), \delta, F)$ is a free crossed module.

In order to recognize elements of $C(\mathcal{Q})$ we need the Identity Property of Reidemeister, [14].

Lemma 4.2. [14], [13] *A word* $w = \prod_i (w_i, r_i)^{\epsilon_i} \in E(\mathcal{Q})$ *represents a Peiffer identity if and only if* w *represents an identity* $W \in I(\mathcal{Q})$ *and there is a pairing* (i, j) *of the indices such that:*

1. $r_i = r_j$
2. $\epsilon_i = -\epsilon_j$, *and*
3. $w_i \langle\langle \underline{r} \rangle\rangle = w_j \langle\langle \underline{r} \rangle\rangle$ *in* $F(\underline{\underline{x}})/\langle\langle r \rangle\rangle_F$

Crossed modules are related to our motivating question by the following well-known theorems.

Theorem 4.3. $\pi_2(X_{\mathcal{Q}}) \cong I(\mathcal{Q})/P(\mathcal{Q})$ *and* $\pi_2(X_{\mathcal{Q}}, X_{\mathcal{Q}}^{(1)}) \cong C(\mathcal{Q})$

Theorem 4.4. [8] *Any free crossed module over a free group has a topological realization as the homotopy crossed module for some two-complex.*

In other words, there is an equivalence between the classification of free crossed modules and the homotopy classification of two-complexes. We can, therefore attempt to illuminate the homotopy classification of two-complexes by studying the generating sets of their second homotopy modules. This can be done by examining their corresponding crossed modules. The Magnus Embedding Theorem provides such an opportunity.

5. The Magnus Embedding Theorem for Second Homotopy Modules

Let G be any finitely presented group. Let $\mathcal{Q} = \langle \underline{\underline{x}} \mid \underline{\underline{r}} \rangle$ be a presentation for the group G, and let $F = F(\underline{\underline{x}})$ be the free group on $\underline{\underline{x}}$. Let $E(\mathcal{Q}) = F(F \times \underline{\underline{r}})$ with non-trivial normal subgroup $P(\mathcal{Q})$ as defined above, and let $C(\mathcal{Q}) = E(\mathcal{Q})/P(\mathcal{Q})$.

Theorem 5.1. *Let M be the free right $\mathbb{Z}(C(\mathcal{Q}))-$module with generators $t_{(w,r)}$ in one-to-one correspondence with the generators of $E(\mathcal{Q})$. Then there exists a map φ so that :*

$$((w,r))\varphi = \begin{pmatrix} P(\mathcal{Q})(w,r) & 0 \\ t_{(w,r)} & 1 \end{pmatrix}$$

and ker $\varphi = (P(\mathcal{Q}))' = [P(\mathcal{Q}), P(\mathcal{Q})]$. Moreover, an element $\prod_i (w_i, r_i)^{\epsilon_i} \in P(\mathcal{Q})$ has image

$$\begin{pmatrix} P(\mathcal{Q})1 & 0 \\ L & 1 \end{pmatrix}$$

where L is a linear combination of $t_{(w,r)}$'s.

The theorem is a particular application of the Magnus Embedding Theorem. For our intentions, we will examine $\pi_2 X_{\mathcal{Q}} \cong I(\mathcal{Q})/P(\mathcal{Q})$. Therefore, we examine the images of these elements to determine generators of $\pi_2 X$ as a right $\mathbb{Z}(C(\mathcal{Q}))$-module. Using the Magnus Embedding Theorem in this situation, we can embed $I(\mathcal{Q})/P(\mathcal{Q})'$ into the matrix group. We know that

$$\pi_2 X_{\mathcal{Q}} \cong I(\mathcal{Q})/P(\mathcal{Q}) \cong \frac{I(\mathcal{Q})/P(\mathcal{Q})'}{P(\mathcal{Q})/P(\mathcal{Q})'}$$

If we examine the image of $I(\mathcal{Q})$ under φ and take the quotient with the image of $P(\mathcal{Q})$ under φ, we can get a representation of $\pi_2 X_{\mathcal{Q}}$. The question then becomes one of recognizing the image of $P(\mathcal{Q})$ under φ.

Theorem 5.2. *Let $W \in E$ and $M = \bigoplus_{(w,r)} t_{(w,r)} \mathbb{Z}C$. Then*

$$W(\varphi) = \begin{pmatrix} P(\mathcal{Q})1 & 0 \\ L & 1 \end{pmatrix}$$

where $L = \sum_i t_{(w_i,r_i)} n_i g_i \in M$ is the image of an element of P if and only if the map

$$\delta_1 : M \to \mathbb{Z}C \text{ defined by } (t_{(w,r)})\delta_1 = (w,r) - 1$$

carries L to 0. Moreover, any $L \in M$ (not necessarily given with its associated matrix) is the image of an element of $P(\mathcal{Q})$ if and only if there exists a pairing (i,j) of indices such that:

(P1) $r_i = r_j$
(P2) $\langle\langle \underline{r} \rangle\rangle_F w_i = \langle\langle \underline{r} \rangle\rangle_F w_j$
(P3) $n_i = -n_j$
(P4) *The map δ_1 carrying $t_{(w_i,r_i)} \mapsto (w_i, r_i) - (1)$ carries L to 0.*

Proof. First we notice that we have a presentation for $C(\mathcal{Q}) = E(\mathcal{Q})/P(\mathcal{Q})$, namely

$$\mathcal{W} = \langle (w,r) \mid (vw^{-1}rw, s)^{-1}(w,r)^{-1}(v,s)(w,r) \rangle$$

for $w, v \in F(\underline{\underline{x}})$, $r, s \in \underline{\underline{r}}$. Therefore, if we look at the augmented chain complex for the universal cover of $X_{\mathcal{W}}$ we have

$$C_2(\widetilde{X_{\mathcal{W}}}) \xrightarrow{\delta_2} C_1(\widetilde{X_{\mathcal{W}}}) \xrightarrow{\delta_1} C_0(\widetilde{X_{\mathcal{W}}}) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where δ_2 is the right Fox derivative and $\left(t_{(w,r)}\right)\delta_1 = \left(\ (w,r) - (1)\ \right) \in C(\mathcal{Q})$ for $(w,r) \in E(\mathcal{Q})$. For ease of notation, let us denote $(vw^{-1}rw, s)^{-1}(w,r)^{-1}(v,s)(w,r)$ by $\langle (w,r),(v,s)\rangle$. Therefore, we have:

$$\begin{array}{ccccccc} \bigoplus_{e_{\langle (w,r),(v,s)\rangle}} \mathbb{Z}C & \xrightarrow{\qquad\delta_2\qquad} & \bigoplus_{e_{(w,r)}} \mathbb{Z}C & \xrightarrow{\delta_1} \mathbb{Z}C & \xrightarrow{\epsilon} \mathbb{Z} & \longrightarrow 0 \\ & \searrow \qquad \nearrow & & & & \\ & H_1(P(\mathcal{Q})) = \ker \delta_1 & & & & \end{array}$$

We are classifying the elements of $H_1(P(\mathcal{Q}))$ which are clearly the ones which are sent to 0 under the δ_1 map. □

6. An Example: A Magnus Embedding Theorem for $\mathbb{Z}_n$

Consider the simple case where $\mathcal{Q} = \langle x \mid x^n \rangle$ presents $\mathbb{Z}_n$. It is not hard to see that the following hold.

Lemma 6.1. *For $\mathcal{Q} = \langle x \mid x^n\rangle$ presenting the group $\mathbb{Z}_n$:*

(1) $F(x) \cong \mathbb{Z}$.
(2) $E(\mathcal{Q}) = F(F(x) \times r)$ *where* $r = x^n$. *Therefore, elements of* $E(\mathcal{Q})$ *are* $(x^i) = (x^i, r)$, *and so* $E(\mathcal{Q}) \cong F(\mathbb{Z})$.
(3) $C(\mathcal{Q}) = E(\mathcal{Q})/P(\mathcal{Q}) \cong \bigoplus_n \mathbb{Z}_{x^i} \cong \mathbb{Z}^n$.

Lemma 6.2. *In this example $I(\mathcal{Q})$ is the normal closure of the element $(x)(1)^{-1}$ in $E(\mathcal{Q})$.*

Proof. The elements which are in $I(\mathcal{Q})$ are the elements which are killed by the map $\delta : E(\mathcal{Q}) \to F(\underline{\underline{x}})$ given by $(w,r)\delta = w^{-1}rw$. Therefore, in our notation, these elements are exactly $(x)(1)^{-1}$:

$$\begin{array}{rcl} \left((x)(1)^{-1}\right)\delta & = & \left((x,r)(1,r)^{-1}\right)\delta \\ & = & x^{-1}x^n x \cdot \left(1^{-1}x^n 1\right)^{-1} \\ & = & x^n x^{-n} \\ & = & 1 \end{array}.$$

□

Lemma 6.3. *In this example, $P(\mathcal{Q})$ is the normal closure in $E(\mathcal{Q})$ of elements of the form*

$$\begin{array}{rcl} (x^{-i}rx^ix^j)^{-1}(x^i)^{-1}(x^j)(x^i) & = & (x^{-i}x^nx^ix^j)^{-1}(x^i)^{-1}(x^j)(x^i) \\ & = & (x^{n+j})^{-1}(x^i)^{-1}(x^j)(x^i) \end{array}$$

for $i, j \in \mathbb{Z}$.

The most interesting part of this example is the following theorem.

Theorem 6.4. *Let $\mathcal{Q} = \langle x \mid x^n \rangle$. Then $E(\mathcal{Q}) = F(\mathbb{Z})$, and the normal closure of the Peiffer elements in $E(\mathcal{Q})$ is a non-trivial normal subgroup. Let $t_{(x^i)}$ be a generating set of a free right $\mathbb{Z}C$-module in one-to-one correspondence with the generators of $E(\mathcal{Q})$. Then there exists a map, φ from $E(\mathcal{Q})$ to a matrix group defined on generators by:*

$$(x^i)\varphi = \begin{pmatrix} P(x^i) & 0 \\ t_{(x^i)} & 1 \end{pmatrix}$$

with kernel $P' = [P, P]$.

□

The more important observation is that we can identify π_2 elements in this embedding.

Lemma 6.5. *Let $\mathcal{Q} = \langle x \mid x^n \rangle$. Elements of $I(\mathcal{Q})$ are in the normal closure of $(x)(1)^{-1}$, and*

$$\left((x)(1)^{-1}\right)\varphi = \begin{pmatrix} P(x)(1)^{-1} & 0 \\ -t_{(1)}(1)^{-1} + t_{(x)}(1)^{-1} & 1 \end{pmatrix}.$$

Theorem 6.6. *Let $\mathcal{Q} = \langle x \mid x^n \rangle$. The image of $P(\mathcal{Q})$ under φ is isomorphic to $P/P' = H_1P$. The images of the Peiffer elements P under φ are*

$$\begin{pmatrix} P1 & 0 \\ L & 1 \end{pmatrix}$$

where $L \in \bigoplus_i t_{(x^i)}\mathbb{Z}C$ is the right Fox derivative, $L = \sum_i a_i t_{x^{k_i}} c_i$ for $c_i \in C, a_i \in \mathbb{Z}$. Moreover, L is the image of a Peiffer element if and only if L has the following generalized identity property:

- (GI1) *The free generators occurring in L satisfy a weak pairing condition: there exists a pairing on the free generators occurring in L, namely $t_{(x^i)} \leftrightarrow t_{(x^j)}$ so that $j \equiv i \pmod{n}$.*
- (GI2) $a_i = -a_j$
- (GI3) $c_i \equiv (x^s)c_j \pmod{P}$ *for some $s \in \mathbb{Z}$.*

Proof. The necessity of the first two conditions follows from the Identity Property. It remains to show that when the pairing condition holds, the coefficients are the same up to a left multiple. This is true in this case because every element of the form (x^{n+j}) can be replaced, modulo elements of P, with $(x^i)(x^j)(x^i)^{-1}$.

Therefore, by making the appropriate substitutions, we can always write the coefficients as products of elements of the form $(x^k)^{\pm 1}$. The result then follows, once again, from the Identity Property. □

It should be possible to get the information provided above to produce the known calculations for $\pi_2 X_{\mathcal{Q}}$. However to date, the computations are complicated and the task of computing over a large group ring is challenging and cumbersome. These calculations are still underway.

We are currently examining examples of more complicated groups, including groups with multiple relators. The calculations become complicated quickly, and we are looking for results which will simplify the calculations required. We believe that this method will allow us to reconstruct Dunwoody's proof of the homotopy inequivalence of the standard two-complexes constructed from presentations of the trefoil knot group, and discover other homotopy inequivalences at non-minimal levels.

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