Projective resolutions from tensoring syzygies

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Abstract

We introduce a method of obtaining a projective bimodule resolution for a quotient Λ of a path algebra by repeatedly tensoring a canonical short exact sequence with $\Omega^1_{\Lambda^e}(\Lambda)$. In one form our method gives a normalized bar resolution used in [4], and we discuss how to modify our construction to obtain a computationally more tractable resolution. We then give a method of computing resolutions of right Λ-modules given in the form of a presentation.

Projective resolutions are a basic tool in studying homological properties of rings and modules. With the introduction of computers computational and algorithmic techniques have grown in importance. In general there is an inverse relationship between the ease in describing the modules and differentials in a resolution and its size. The bar resolution (see [10] for example) is quite easily described but is often too large for computations. In our context there is a unique minimal resolution, but it is often quite difficult to describe. A compromise is to use a 'normalized bar resolution', see [4] or [6], smaller in size but still affording an easy description. The techniques in this paper will arrive at such a resolution in a different way, and then we will discuss a way to modify our construction to make the differentials less computationally intensive. As always there will be a trade off in a less easy description of these differentials.

The algebras studied in this paper are quotients of path algebras. Let $Q = (Q_0, Q_1)$ be a quiver, a directed graph with possibly multiple arrows between vertices, where Q_0 denotes the vertex set and Q_1 the arrow set. For a field K the path algebra KQ has K-basis consisting of the set of all finite directed paths in Q, where we consider a vertex a path of length zero. Each path has an origin vertex $o(p)$ and a terminal vertex $t(p)$, with a vertex path being its own origin and terminus. We multiply two paths p and q by concatenation pq if $t(p) = o(q)$, and the product is zero otherwise. If I is an ideal in KQ we let $\Lambda = KQ/I$. The class of algebras which are quotients of path algebras include all finitely generated associative K-algebras. Any finite dimensional K-algebra where K is algebraically closed is Morita equivalent to a quotient of a path algebra by an *admissible* ideal I, that is $J^2 \subset I \subset J^N$ for some N where J is the ideal in KQ generated by Q_1. Also we have graded K-algebras $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \ldots$ where Λ_0 is a finite direct sum of copies of K, Λ_i is finite dimensional over K, and Λ is generated in degrees 0 and 1.

For $\Lambda = KQ/I$ we let Λ_0 denote KQ_0 (whether Λ is graded or not). Then $\Lambda \cong \Lambda_0 \oplus \mathbf{r}$ as a Λ_0-Λ_0-bimodule, where $\mathbf{r}$ is $\mathrm{rad}(\Lambda) = \sum_{a \in Q_1} a\Lambda$. Associated to each vertex $v \in Q_0$ is an indecomposable projective right Λ-module $v\Lambda$, and these vertex projectives form a complete set of representatives of the isomorphism classes of indecomposable projective Λ-modules. Each projective right Λ-module then has the form $L \otimes_{\Lambda_0} \Lambda$ for some right Λ_0-module L. Projective left Λ-modules are $\Lambda \otimes_{\Lambda_0} L$ for L a left Λ_0-module. Let Λ^e denote the enveloping algebra of Λ, $\Lambda^{op} \otimes \Lambda$ (here, as it does throughout the paper, $\otimes$ denotes $\otimes_K$). The category of right Λ^e modules is equivalent to the category of Λ-Λ-bimodules, and we will not distinguish between these two categories. In $\mathrm{mod}(\Lambda^e)$ the indecomposable projectives are $\Lambda v \otimes w\Lambda$, and all projectives are $\coprod \Lambda v \otimes w\Lambda = \Lambda \otimes_{\Lambda_0} L \otimes_{\Lambda_0} \Lambda$ for some Λ_0-Λ_0-module L.

We will begin with the short exact sequence $\pi = 0 \to \Omega \xrightarrow{\imath} P^0 \xrightarrow{m} \Lambda \to 0$ of Λ^e-modules, where $P^0 = \coprod_v \Lambda v \otimes v\Lambda = \Lambda \otimes_{\Lambda_0} \Lambda$ and Ω denotes $\Omega^1_{\Lambda^e}(\Lambda)$. It is well known that Ω is projective as a right Λ-module, so the functor $\Omega \otimes_\Lambda$ _ is exact. Thus $\Omega \otimes_\Lambda \pi$ is again a short exact sequence, as is $\Omega^{(n)} \otimes_\Lambda \pi$ where $\Omega^{(n)}$ denotes $\Omega^{\otimes_\Lambda n}$. We will see that the module $\Omega^{(n)} \otimes_\Lambda P^0$ is projective as a Λ-Λ-bimodule, and so we can splice our short exact sequences together using the obvious isomorphism $\Omega^{(n)} \otimes_\Lambda \Lambda \to \Omega^{(n-1)} \otimes_\Lambda \Omega$ to obtain a projective resolution of Λ as a Λ^e-module. Bimodule resolutions of Λ are important to the study of Hochschild cohomology. While the modules in the minimal Λ^e-resolution are given by Happel in [9], the maps are known only for monomial algebras, due to Bardzell [3]. In the next section we describe the bimodule resolution of Λ obtained using the method given above, and it turns out to be a normalized bar resolution. In Section 2 we alter our construction to produce a resolution with the same projective modules, but with computationally less complex differentials. Following this we give a method of obtaining right-module projective resolutions for Λ-modules M_Λ given in the form of a projective presentation, which is essentially tensoring our bimodule resolution over Λ with a syzygy of M. Tensoring a bimodule resolution over Λ to obtain a one sided resolution of a module M goes back to [2]. What is notable about our construction is that it gives a method to compute resolutions which is not recursive, in that one need not compute all prior projectives in order to obtain the n-th term in a resolution. Current methods require such computations, with the price of computing a new Gröbner basis at each step.

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1 The resolution

Consider again the short exact sequence of Λ-Λ-bimodules $\pi = 0 \to \Omega \stackrel{\imath}{\to} P^0 \stackrel{m}{\to} \Lambda \to 0$. We will splice together the short exact sequences $\Omega^{(n)} \otimes_\Lambda \pi = 0 \to \Omega^{(n)} \otimes_\Lambda \Omega \to \Omega^{(n)} \otimes_\Lambda P^0 \to \Omega^{(n)} \otimes_\Lambda \Lambda \to 0$ to obtain a projective resolution of Λ. That $P^n = \Omega^{(n)} \otimes_\Lambda P^0$ is projective follows from the next proposition, which is a slight generalization of a result in [2], where it is proved in the case that ${}_\Lambda M_\Lambda$ is projective as both a left and right Λ-module. We do not need the extra generality here, but we include our proof as it hints at complications that are to come in determining the maps in our resolution.

Proposition 1.1. *Let ${}_\Lambda M_\Lambda$ be a Λ-Λ-bimodule which is projective as a left Λ-module. Then for any projective Λ^e module P we have that $M \otimes_\Lambda P$ is a projective Λ^e module.*

Proof. It suffices to show that $M \otimes_\Lambda (\Lambda v \otimes w\Lambda)$ is a projective bimodule. We note that $M \otimes_\Lambda (\Lambda v \otimes w\Lambda)$ is clearly isomorphic to $Mv \otimes w\Lambda$. Since M is projective as a left Λ-module there exists ${}_\Lambda N$ so that $M \oplus N$ is free as a left Λ-module. We note that since v is an idempotent, $Mv \oplus M(1-v) \cong M$, whence $Mv \oplus (M(1-v) \oplus N)$ is free, and Mv is still projective as a left Λ-module. Thus $Mv \cong \coprod \Lambda u$ where u is a vertex in Γ_0, and hence $M \otimes_\Lambda (\Lambda v \otimes w\Lambda) \cong Mv \otimes w\Lambda \cong \coprod(\Lambda u \otimes w\Lambda)$, which is clearly a Λ^e projective module. ☐

The complications that are to come in determining the maps in our resolution are involved in viewing $\Omega^{(n)}$ as a projective left Λ-module, and having to express right multiples of elements in $\Omega^{(n)}$ in terms of the left Λ-module representation, as we did when we expressed Mv as $\coprod \Lambda u$. First we discuss the structure of Ω and how to do this when $n=1$. Recall that $P^0 = \Lambda \otimes_{\Lambda_0} \Lambda$ and that $\Lambda \cong \Lambda_0 \oplus \mathbf{r}$, so as a left Λ-module $P^0 = (\Lambda \otimes_{\Lambda_0} \Lambda_0) \oplus (\Lambda \otimes_{\Lambda_0} \mathbf{r})$. One readily sees that under the multiplication map $m : P^0 \to \Lambda$ restricting to the first summand gives an isomorphism, so the kernel Ω will be isomorphic to $\Lambda \otimes_{\Lambda_0} \mathbf{r}$ as a left Λ-module. We can explicitly describe the inclusion map $\imath : \Omega \to P^0$ by $\imath(\lambda \otimes_{\Lambda_0} r) = \lambda r \otimes_{\Lambda_0} t(r) - \lambda \otimes_{\Lambda_0} r$ (here we are viewing P^0 in its bimodule form $\Lambda \otimes_{\Lambda_0} \Lambda$). The next proposition tells us how to right multiply elements of Ω in their left module representation.

Proposition 1.2. *Let $\omega = \lambda \otimes_{\Lambda_0} r \in (\Lambda \otimes_{\Lambda_0} \mathbf{r}) \cong \Omega$. If $v \in \Lambda_0$ then $\omega \cdot v = \omega$ if and only if $t(r) = v$, and is zero otherwise. If $x \in \mathbf{r}$ then $\omega \cdot x = \imath^{-1}(\imath(\omega) \cdot x) = \lambda \otimes_{\Lambda_0} rx - \lambda r \otimes_{\Lambda_0} x$ in our $\Lambda \otimes_{\Lambda_0} \mathbf{r}$ representation of Ω.*

Proof. Since $\imath$ is monic, we may identify $\omega = \lambda \otimes_{\Lambda_0} r$ in $\Lambda \otimes_{\Lambda_0} \mathbf{r}$ with $\imath(\omega) = \lambda r \otimes_{\Lambda_0} t(r) - \lambda \otimes_{\Lambda_0} r$ in $P^0 = \Lambda \otimes_{\Lambda_0} \Lambda$. If $v \in \Lambda_0$ the result is now obvious. For $x \in \mathbf{r}$ we note that $\imath(\omega) \cdot x$ is equal to $\lambda r \otimes_{\Lambda_0} x - \lambda \otimes_{\Lambda_0} rx$. Note that $\imath(\lambda \otimes_{\Lambda_0} rx - \lambda r \otimes_{\Lambda_0} x) = \lambda r \otimes_{\Lambda_0} x - \lambda \otimes_{\Lambda} rx = \imath(\omega) \cdot x$. The result now follows. $\square$

We now turn our attention to $\Omega^{(n)}$. In the proof of the next proposition we are going to be quite explicit as this will pay dividends when we return to look at right multiplication of left module representations of elements of $\Omega^{(n)}$ by elements of $\mathbf{r}$.

Proposition 1.3. *As a left Λ-module, $\Omega^{(n)} \cong \Lambda \otimes_{\Lambda_0} \mathbf{r}^{\otimes_{\Lambda_0} n}$.*

Proof. Since $\Omega \cong \Lambda \otimes_{\Lambda_0} \mathbf{r}$ we have $\Omega^{(n)} \cong (\Lambda \otimes_{\Lambda_0} \mathbf{r})^{\otimes_\Lambda n}$. A basic tensor then takes the form $\omega = \lambda_1 \otimes_{\Lambda_0} r_1 \otimes_\Lambda \lambda_2 \otimes_{\Lambda_0} r_2 \otimes_\Lambda \ldots \otimes_\Lambda \lambda_n \otimes_{\Lambda_0} r_n$. We are free to pull the λ_i to the left hand side of the $\otimes_\Lambda$s, so we have $\omega = \lambda_1 \otimes_{\Lambda_0} r_1\lambda_2 \otimes_\Lambda t(\lambda_2) \otimes_{\Lambda_0} r_2\lambda_3 \otimes_\Lambda t(\lambda_3) \otimes_{\Lambda_0} \ldots \otimes_{\Lambda_0} r_{n-1}\lambda_n \otimes_\Lambda t(\lambda_n) \otimes_{\Lambda_0} r_n$. Of course one now sees how to describe the isomorphism taking ω to $\lambda_1 \otimes_{\Lambda_0} r_1\lambda_2 \otimes_{\Lambda_0} r_2\lambda_3 \otimes_{\Lambda_0} \ldots \otimes_{\Lambda_0} r_{n-1}\lambda_n \otimes_{\Lambda_0} r_n$, and we have established the result. $\square$

More importantly an element $\lambda \otimes_{\Lambda_0} s_1 \otimes_{\Lambda_0} s_2 \otimes_{\Lambda_0} \ldots \otimes_{\Lambda_0} s_n = \lambda(s_1, \ldots, s_n)$ of $\Omega^{(n)} \cong \Lambda \otimes_{\Lambda_0} \mathbf{r}^{\otimes^n_{\Lambda_0}}$, corresponds to $\lambda \otimes_{\Lambda_0} s_1 \otimes_\Lambda t(s_1) \otimes_{\Lambda_0} s_2 \otimes_\Lambda t(s_2) \otimes_{\Lambda_0} \ldots \otimes_{\Lambda_0} s_n$, in $(\Lambda \otimes_{\Lambda_0} \mathbf{r})^{\otimes^n_\Lambda}$; that is, except for the first, each Λ piece is in Λ_0. Now if we consider $\omega = \lambda(r_1, \ldots, r_n) \in \Omega^{(n)}$ in our left module description $\Lambda \otimes_{\Lambda_0} \mathbf{r}^{\otimes_{\Lambda_0} n}$ and want to express $\omega \cdot x$ in a similar form we are ready to do this by repeatedly appealing to Proposition 1.2.

Proposition 1.4. *For* $v \in \Lambda_0$ *and* $\omega = \lambda(r_1, r_2, \ldots, r_n) \in \Omega^{(n)}$, $\omega \cdot v = \omega$ *if and only if* $t(r_n) = v$, *and is zero otherwise. For* $x \in \mathbf{r}$, $\omega \cdot x = \lambda(r_1, \ldots, r_{n-1}, r_n x) + \sum_{i=1}^{n-1}(-1)^{n-1}\lambda(r_1, \ldots, r_i r_{i+1}, r_{i+2}, \ldots, r_n, x) + (-1)^n \lambda r_1(r_2, r_3, \ldots, r_n, x)$.

Proof. Recall $\lambda(r_1, \ldots r_n)$ stands for $\lambda \otimes_{\Lambda_0} r_1 \otimes_\Lambda t(r_1) \otimes_{\Lambda_0} \ldots \otimes_{\Lambda_0} r_n$ in $(\Lambda \otimes_{\Lambda_0} \mathbf{r})^{\otimes_\Lambda n}$ (each Λ piece is in Λ_0). We first need to multiply $t(r_{n-1}) \otimes_{\Lambda_0} r_n$ in Ω by x, and applying Proposition 1.2 we have $t(r_{n-1}) \otimes_{\Lambda_0} r_n \cdot x = t(r_{n-1}) \otimes_{\Lambda_0} r_n x - r_n \otimes_{\Lambda_0} x$. To express $\lambda(r_1, \ldots, r_{n-1}, r_n \cdot x)$ in our $\mathbf{r}^{\otimes_{\Lambda_0} n}$ notation we must pull the r_n to the left hand side of the $\otimes_\Lambda$ in the second term, so now we have $\omega \cdot x = \lambda(r_1, \ldots, r_n x) - \lambda(r_1, \ldots, r_{n-1} \cdot r_n, x)$. We must apply Proposition 1.2 again, this time in the second $\Lambda \otimes_{\Lambda_0} \mathbf{r}$ term from the left, and continue inductively obtaining the result. □

We are now ready to describe our resolution beginning with a description of the projectives.

Proposition 1.5. *The projective* Λ^e*-module* $\Lambda u \otimes v \Lambda$ *occurs with multiplicity* $dim_K\,(u\,\mathbf{r}^{\otimes_{\Lambda_0} n}\,v)$ *in the indecomposable decomposition of* P^n.

Proof. Since $\Omega^{(n)} \cong \Lambda \otimes_{\Lambda_0} rad^{\otimes_{\Lambda_0} n}$, we have $\Omega^{(n)} \otimes_\Lambda \Lambda v \otimes v\Lambda \cong \Lambda \otimes_{\Lambda_0} \mathbf{r}^{\otimes_{\Lambda_0} n} \otimes_\Lambda v \otimes v\Lambda$. Taking the direct sum over all $v \in Q_0$ the result now follows. □

Our goal now is to understand the maps $d_n : P^n = \Omega^{(n)} \otimes_\Lambda P^0 \to P^{n-1} = \Omega^{(n-1)} \otimes_\Lambda P^0$ in our resolution. As $P^n \cong \coprod \Lambda u \otimes v \Lambda$ the map d_n is completely determined by its action on the elements $u \otimes v$ comprising $\mathrm{Top}(P^n)$. Thus we need only compute $d_n(u \otimes v)$, where $u \otimes v$ is identified with a non-zero tensor $u \otimes_{\Lambda_0} (r_1, \ldots, r_n) \otimes_\Lambda v \otimes v$ in $\Omega^{(n)} \otimes_\Lambda P^0$. Now d_n is really a composition of three maps, the first map is the map on the right hand side of the short exact sequence $\Omega^{(n)} \otimes_\Lambda \pi$, which we note leaves the $u \otimes_{\Lambda_0} (r_1, \ldots, r_n)$ part fixed, and sends $v \otimes v$ to v. The second map is the multiplication isomorphism $\Omega^{(n)} \otimes_\Lambda \Lambda \to \Omega^{(n-1)} \otimes_\Lambda \Omega$, and we have $u \otimes v$ mapping to $u(r_1, \ldots, r_n) \cdot v = u(r_1, \ldots, r_n)$. To be clear about the $\Omega^{(n-1)} \otimes_\Lambda \Omega$ structure we will rewrite this $u(r_1, \ldots r_{n-1}) \otimes_\Lambda t(r_{n-1}) \otimes_{\Lambda_0} r_n$. Finally we will have to apply the map on the left hand side of $\Omega^{(n-1)} \otimes_\Lambda \pi$, which is the identity on $n-1$ copies of Ω and the inclusion map $\imath$ on the last term. We have $\imath(t(r_{n-1}) \otimes_{\Lambda_0} r_n) = r_n \otimes_{\Lambda_0} t(r_n) - t(r_{n-1}) \otimes_{\Lambda_0} r_n$, so under $\mathrm{id}_{\Omega^{(n-1)}} \otimes_\Lambda \imath$ we see that $u(r_1, \ldots, r_{n-1}) \otimes_\Lambda t(r_n) \otimes_{\Lambda_0} r_n$

maps to $u(r_1, \ldots, r_{n-1}) \cdot r_n \otimes_{\Lambda_0} v - u(r_1, \ldots, r_{n-1}) \cdot t(r_{n-1}) \otimes_{\Lambda_0} r_n$. Applying Proposition 1.4 and letting $u(r_1, \ldots, r_n)v$ denote $u(r_1, \ldots, r_n)v \otimes v$ we have

$$d_n(u(r_1, \ldots, r_n)v) = -u(r_1, \ldots, r_{n-1})r_n + \sum_{i=1}^{n-2}(-1)^{n-1-i}(r_1, \ldots, r_i r_{i+1}, \ldots, r_n)v + (-1)^{n-1} r_1(r_2, \ldots, r_n),$$

and we have exhibited the normalized bar resolution.

2 Reducing the Complexity of d_n

In this section we use an alternate form of the inclusion map $\imath$, which we will denote $\imath'$, to modify the results on right multiplication of Ω in the previous section. Of course we are not really producing a 'new' right multiplication; the fact is that using a different inclusion map $\imath'$ makes the element $\lambda \otimes_{\Lambda_0} r$ of $\Lambda \otimes_{\Lambda_0} \mathbf{r}$ stand for something different in $\Omega \subset P^0$, and hence the same old right multiplication takes on a new form, since we multiply $\omega \cdot x = \imath^{-1}(\imath(\omega) \cdot x)$. Recall that in the previous section the differentials involved multiplying an element of $\Omega^{(n-1)}$ on the right by something in $\mathbf{r}$, and this 'new' multiplication will result in a computationally less intensive way to do this, hence a computationally less intensive differential.

We note that the set B of all oriented paths in KQ forms a K-basis, and we may choose a basis for a vector space complement of I containing both Q_0 and Q_1, which we'll denote N. We set N_+ to be $N \setminus Q_0$ and note that N_+ is a basis for $\mathbf{r}$. In order to do this in a 'nice' way one may choose a well order $<$ on the elements of B which is *admissible*, meaning that $\alpha < \beta$ implies $\alpha\gamma < \beta\gamma$ and $\gamma\alpha < \gamma\beta$ as long as both products we are comparing are non-zero. One example of such an order is the length-lex order: arbitrarily order Q_0 and Q_1, and for paths p and q of length at least two set $p < q$ if $len(p) < len(q)$, or if $len(p) = len(q)$ and p comes before q in the 'dictionary' using the chosen order on the arrows (letters). A variation on the commutative Buchberger algorithm allows us to compute a special generating set $\mathcal{G}$ for I called a reduced Gröbner basis, and obtain a subset $Nontip(I)$ $(= N)$ of B consisting of those paths in B which are not divisible by the largest term (called a *tip*) of any element of $\mathcal{G}$. Multiplication may now be done using Gröbner reduction; any time a product of two paths pq produces something not in N_+ we appeal to Gröbner reduction to rewrite

pq as a linear combination of elements of N_+. Note that when considering $\lambda(r_1, \dots r_n) \in \Omega^{(n)}$ as we did above we need only consider $r_i \in N_+$.

For a path p in N_+ let $f(p)$ be the first arrow in the path, and let p^- be what remains, which may be only a vertex (in Λ_0). Thus $p = f(p) \cdot p^-$. Since Ω as a sub-bimodule of $P^0 = \Lambda_0 \otimes_{\Lambda_0} \Lambda_0$ is generated by $a \otimes_{\Lambda_0} t(a) - o(a) \otimes_{\Lambda_0} a$ ($a \in Q_1$) one readily sees that we may take our inclusion map $\imath' : \Lambda \otimes_{\Lambda_0} \mathbf{r} \to \Lambda \otimes_{\Lambda_0} \Lambda$ by $\imath'(\lambda \otimes_{\Lambda_0} r) = \lambda f(r) \otimes_{\Lambda_0} r^- - \lambda \otimes_{\Lambda_0} r$ (recall r may be considered a path in N_+). Our right multiplication in Ω still takes the form $(\omega) \cdot x = {\imath'}^{-1}(\imath'(\omega) \cdot x)$, but with the new $\imath'$ in place of $\imath$ we have $(\lambda \otimes_{\Lambda_0} r) \cdot x = \lambda \otimes_{\Lambda_0} rx$ as long as $rx \in N_+$. If $rx \notin N_+$ there is some $g = a_1 \cdots a_m + \sum_i k_i (a_{i,1} \cdots a_{i,m_i})$ in $\mathcal{G}$ with $a_1 \cdots a_m$ dividing rx (here we are assuming $a_1 \cdots a_m = tip(g)$, and each of the a_i's are arrows in Q_1). If $f(a_1 \dots a_m) = a_1 \neq f(rx) = f(r)$ we apply Gröbner reduction as usual to write $\lambda \otimes_{\Lambda_0} rx$ as a sum $\sum_j k_j \lambda \otimes_{\Lambda_0} n_j$ for some $n_j \in N_+$. The complication arises if $a_1 = f(r)$.

Proposition 2.1. *Using the notation above, if $rx = a_1 \cdots a_m$ then we multiply $(\lambda \otimes_{\Lambda_0} r) \cdot x = {\imath'}^{-1}(\lambda \imath(o(r) \otimes_{\Lambda_0} r) \cdot x)$ which is $(\sum_{j=2}^{m} -\lambda a_1 \cdots a_{j-1} \otimes_{\Lambda_0} a_j \cdots a_m) - (\sum_i k_i \sum_{j=1}^{m_i} a_{i,1} \cdots a_{i,j-1} \otimes_{\Lambda_0} a_{i,j} \cdots a_{i,m_i})$.*

Proof. We see that

$$\begin{aligned} a_1 \cdots a_m \otimes_{\Lambda_0} t(g) - o(g) \otimes_{\Lambda_0} a_1 \cdots a_m = \\ \sum_{j=1}^{m} a_1 \cdots a_j \otimes_{\Lambda_0} a_{j+1} \cdots a_m - a_1 \cdots a_{j-1} \otimes_{\Lambda_0} a_j \cdots a_m = \\ \sum_{j=1}^{m} a_1 \cdots a_{j-1} \cdot \imath'(o(a_j) \otimes_{\Lambda_0} a_j \cdots a_m). \end{aligned}$$

Note that we have changed to *left* multiplication. Similarly

$$\begin{aligned} a_{i,1} \cdots a_{i,m_i} \otimes_{\Lambda_0} t(g) - o(g) \otimes_{\Lambda_0} a_{i,1} \cdots a_{i,m_i} = \\ \sum_{j=1}^{m_i} a_{i,1} \cdots a_{i,j-1} \cdot \imath'(a_{i,j} \cdots a_{i,m_i}). \end{aligned}$$

We observe that

$$
\begin{aligned}
& a_1 \cdots a_m \otimes_{\Lambda_0} t(g) - o(g) \otimes_{\Lambda_0} a_1 \cdots a_m + \\
& \qquad \sum_i k_i a_{i,1} \cdots a_{i,m_i} \otimes_{\Lambda_0} t(g) - o(g) \otimes_{\Lambda_0} a_{i,1} \cdots a_{i,m_i} \\
& = \sum_{j=1}^{m} a_1 \cdots a_{j-1} \cdot i'(o(a_j) \otimes_{\Lambda_0} a_j \cdots a_m) + \\
& \qquad \sum_i k_i \sum_{j=1}^{m_i} a_{i,1} \cdots a_{i,j-1} \cdot i'(o(a_{i,j}) \otimes_{\Lambda_0} a_{i,j} \cdots a_{i,m_i}) \\
& = g \otimes_{\Lambda_0} t(g) - o(g) \otimes_{\Lambda_0} g.
\end{aligned}
$$

We now solve for $o(a_1) \otimes_{\Lambda_0} a_1 \cdots a_m$, and the result follows. □

The important thing to take away from all this is that right multiplication of $o(r) \otimes_{\Lambda_0} r$ in Ω does not always result in terms with a positive length path on the left, as it did in Section 1 where $(o(r) \otimes_{\Lambda_0} r) \cdot x$ became $r \otimes_{\Lambda_0} x - o(r) \otimes_{\Lambda} rx$. Here as long as rx is not a *tip*, $(o(r) \otimes_{\Lambda_0} r) \cdot x = o(r) \otimes_{\Lambda_0} rx$. This formula contains more detail due to our use of an explicit basis B, rather than our earlier basis-free notation. Unlike KQ with basis B, a basis for $\mathbf{r}$ may not be multiplicative in the sense that $b_1 \cdot b_2 \in B$ for $b_i \in B$. In general the product $b_1 \cdot b_2$ remains in $\mathbf{r}$, but not necessarily in B, so best one may say is that we may express $b_1 \cdot b_2$ as a linear combination of basis elements. Therefore writing $(r_1, \ldots, r_i r_{i+1}, \ldots, r_n)$ as we did in the formula for d_n in the previous section belies the fact that the product $r_i r_{i+1}$ may indeed have many terms in it.

We can use our 'new' right multiplication and the new i' to construct a new differential d' on the projectives in the complex in Section 1. We follow the discussion after Proposition 1.5 and the key becomes right multiplying the $t(r_{n-2}) \otimes_{\Lambda_0} r_{-1}$ by r_n, and inductively pushing ones way back in the sequence $(r_1, \ldots, r_{n-1})$. The reader should observe that using the new multiplication will result in a savings of effort when computing the differential in our resolution.

As an example suppose that $\Lambda = k[x]/x^3$. Here Q consists of one vertex v and one arrow x, $\mathcal{G} = \{x^3\}$, $N = \{v, x, x^2\}$ with $N_+ = \{x, x^2\}$. If we consider the map from P^4 to P^3 for example we find that using the right multiplication in Section 1 we have $d_4(v(x,x,x,x)v \otimes v) = -v(x,x,x)x + v(x,x,x^2)v - v(x,x^2,x)v + v(x^2,x,x)v - x(x,x,x)v$. Using our new right multiplication to construct the differential, which we'll denote d', we have $d'_4(v(x,x,x,x)v \otimes v) = -v(x,x,x)x + v(x,x,x^2)$. Note that there was no need to 'push back' any further using this right multiplication.

3 Resolutions of Modules Given By Presentations

We assume now that we have a right Λ-module M given in the form of a projective presentation, $M \cong \operatorname{coker}(Q^1 \xrightarrow{f} Q^0)$. Here we let $Q^1 = \coprod_T w_t\Lambda$ and $Q^0 = \coprod_S v_s\Lambda$, and the data we assume we have is f in the form of a 'matrix' (f_{st}) with $f_{st} \in v_s\Lambda w_t$, which we identify with $\operatorname{Hom}_\Lambda(w_t\Lambda, v_s\Lambda)$. Given such a matrix, along with the quiver Q, one can easily determine the v_s and w_t. We are interested in projective resolutions and homology of M, but to compute the $(n+1)$-st homology $\operatorname{Ext}^{n+1}_\Lambda(M,N)$ or $\operatorname{Tor}^\Lambda_{n+1}(M,N)$ we need only $Q^{n+2} \to Q^{n+1} \to Q^n$ in a projective resolution of M. Current methods, in [5] or [8] for example, require that one compute each Q^i with $i < n$ to obtain Q^n and back, and each step requires the computation of a new Gröbner basis, which is computationally expensive. In this section we give a method for computing $Q^{n+2} \to Q^{n+1} \to Q^n$ without computing the previous portion of the resolution, and requiring that we begin with only a Gröbner basis for I. Inductively one could continue back in the resolution from this point computing Q^{n+3} and so on, but the point is that we are able to skip computing Q^i for $i < n$ if we choose. Our method basically involves taking a Λ^e resolution and tensoring over Λ with a syzygy of M, producing a Λ-resolution, and we refer the reader to [2] for motivation and details.

Before we begin in earnest we will need to understand how projective right Λ-modules tensor with the projectives P^n in our resolution. The proof of the next lemma comes directly from the already given structure of P^n, and is left to the reader.

Lemma 3.1. *Let $R = \coprod_u u\Lambda$ be a projective right Λ-module. Then $R \otimes_\Lambda P^n \cong \coprod_u u\Lambda \otimes_{\Lambda_0} \mathbf{r}^{\otimes_{\Lambda_0} n} \otimes_{\Lambda_0} \Lambda$, and the indecomposable projective $e\Lambda$ occurs with multiplicity $\sum_u dim_K(u\Lambda \otimes_{\Lambda_0} \mathbf{r}^{\otimes_{\Lambda_0} n} \otimes_{\Lambda_0} e)$. Note that a K-basis for $\operatorname{Top}(Q \otimes_\Lambda P^n)$ can be associated to a set of all sequences $(u, \lambda, r_1, \ldots, r_n)$ where u is one of the vertices in Q_0 determining R, $\lambda \in u\Lambda$, $r_i \in N_+$ for all i, and $\lambda \otimes_{\Lambda_0} r_1 \otimes_{\Lambda_0} \ldots \otimes_{\Lambda_0} r_n \neq 0$.* □

We will henceforth identify the sequence $(u, \lambda, r_1, \ldots, r_n)$ with the corresponding element of $\mathrm{Top}(R \otimes_\Lambda P^n)$. To see where we are going consider the following commutative diagrams:

$$\begin{array}{ccc} Q^1 \otimes_\Lambda P^{n+1} & \to & Q^1 \otimes_\Lambda P^n \\ \downarrow & & \downarrow \\ Q^0 \otimes_\Lambda P^{n+1} & \to & Q^0 \otimes_\Lambda P^n \end{array} \qquad \begin{array}{ccc} Q^1 \otimes_\Lambda P^{n+1} & & Q^1 \otimes_\Lambda P^n \\ \downarrow & & \downarrow \\ \Omega^1_\Lambda(M) \otimes_\Lambda P^{n+1} & \to & \Omega^1_\Lambda(M) \otimes_\Lambda P^n \\ \downarrow & & \downarrow \\ Q^0 \otimes_\Lambda P^{n+1} & & Q^0 \otimes_\Lambda P^n. \end{array}$$

The right diagram is of course not exact, but we have highlighted the actual image of the vertical maps in the left diagram in the modules in the center on the right. From the previous lemma we are able to determine the structure of the Top of each of the four projectives in the left diagram. We will discuss how to compute the image of the vertical maps, which we notice is $\Omega^1_\Lambda(M) \otimes_\Lambda P^i$, and we will then be able to compute the Top of these projectives. Using the bottom horizontal map in the left diagram we will compute its restriction to $\Omega^1_\Lambda(M) \otimes_\Lambda P^{n+1} \to \Omega^1_\Lambda(M) \otimes_\Lambda P^n$, which becomes the horizontal map in the diagram on the right. By splicing together these horizontal maps we will be able then to compute a projective resolution of M beginning anywhere we wish.

The first thing to do then is to compute $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P^n)$. By the previous Lemma we have the indecomposable projectives in $Q^1 \otimes_\Lambda P^n$ corresponding to sequences $(w_t, \lambda, r_1, \ldots, r_n)$. Elements of $\mathrm{Top}(Q^0 \otimes_\Lambda P^n)$ correspond to sequences $(v_s, \lambda', r'_1, \ldots, r'_n)$. Applying $f \otimes_\Lambda \mathrm{id}_{P^n}$ to the element $(w_t, \lambda, r_1, \ldots, r_n)$ we get $\sum_s (v_s, f_{st}\lambda, r_1, \ldots, r_n)$ since the w_t and the λ correspond to the Q^1 part of $Q^1 \otimes_\Lambda P^n$, and the vertical map is the identity on $\Omega^{(n)}$. Notice that the image remained in $\mathrm{Top}(Q^0 \otimes_\Lambda P^n)$. Doing this for each $(w_t, \lambda, r_1, \ldots, r_n)$ we obtain a vector space spanning set for $\mathrm{Im}(f \otimes_\Lambda \mathrm{id}_{P^n}) = \mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P^n)$ and from this one can compute a K-basis for $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P^n)$. So we are able to determine the indecomposable projective direct summands of $\Omega^1_\Lambda(M) \otimes_\Lambda P^i$, for the price of one row reduction each. Using our resolution from Section 1 or 2 we can easily compute the horizontal map $Q^0 \otimes_\Lambda P^{n+1} \to Q^0 \otimes_\Lambda P^n$ in terms of our bases $(v_s, \lambda, r'_1, \ldots, r'_{n+1})$, and we have to compute a change of basis to express the image of an element in $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P^{n+1})$ as a sum of right Λ-multiples of elements of $\mathrm{Top}(\Omega^1_\Lambda(M) \otimes_\Lambda P^n)$. Thus beginning with a Gröbner basis $\mathcal{G}$

for I, row reducing three times, and computing two change of bases, we are able to produce $Q^{n+2} \to Q^{n+1} \to Q^n$.

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