Non-Attractors of Iterated Function Systems

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Abstract: An example of a locally connected continuum which is not the attractor of any iterated function system (IFS) in $\mathbb{R}^2$ is constructed in a work of Kwieciński [4]. The example is relative to a question of Hata [3] regarding the existence of such continua. Kwieciński points out that a variation on his main construction provides an arc in $\mathbb{R}^2$ which is not an attractor of any IFS. Here, criteria are developed that pertain to ascertaining whether or not a given arc embedded in Euclidean n-space can be realized as an attractor of some IFS. Besides reaffirming the example in [4], the characterization is sufficiently strong so as to provide many examples of arcs in $\mathbb{R}^n\,(n \geq 2)$ which are not attractors of any IFS. Moreover, the techniques developed show that many arcs (as well as other compacta) in Euclidean spaces are indeed attractors of iterated function systems; techniques to construct such systems are developed naturally along the way. As a final note, there are arcs in $\mathbb{R}^n$ that fail to meet the eligibility requirements for application of the developed criteria. Several open questions and conjectures regarding this issue will be discussed.

1 Introduction

If $X = X_d$ is a metric space, a contraction map on X is a function $f : X \to X$ with the feature that $d(f(x), f(y)) \leq r \cdot d(x, y)$ where $0 \leq r < 1$ for each $x, y \in X$. It is easily verified that such a function is necessarily continuous so that the word map is used in the usual way. A (hyperbolic) iterated function system (IFS) consists of a complete metric space X together with a finite set of contraction maps on X. There is a unique compact set associated with a given IFS called the attractor of the IFS. Barnsley [1] popularized the notion of iterated function systems by revealing their ability to encode visual images as attractors.

Mathematically, there remain open questions about IFSs and their attractors. While it is possible to approximate any compact subset in the space X by an attractor of some IFS [1], the question as to which compacta can be realized as attractors of IFSs remains elusive. Kwieciński [4] gives an example of a locally connected continuum which cannot be realized as an attractor of any IFS in $\mathbb{R}^2$. A variation on his main argument provides an example of an arc which is not an attractor of any IFS as well. While Hata [3] showed that a connected attractor must be locally conected, he went on to pose the question of whether or not each locally connected continuum in $\mathbb{R}^n$ could be realized as the fixed point set of a finite collection of *weak* contractions. Kwieciński's result pertains to this question by showing that, indeed, there are locally connected continua continua which are not realizeable as attractors of IFSs.

In this work, we shall develop techniques to point out that many arcs in $\mathbb{R}^2$ (and in $\mathbb{R}^n$ in general) are not attractors of any IFS by examining a characteristic pertaining to a particular embedding of $[0,1]$ into $\mathbb{R}^n$. This characterization will be used to develop an applicable criterion to attempt to detect whether or not a given arc is an attractor. Kwieciński's example will be reaffirmed by this criterion. Moreover, we will be able to easily produce many more examples of locally connected continuua which are not attractors. Also of note, we shall prove that many arcs are indeed attractors and provide techniques to construct their corresponding IFSs. We'll continue by observing that there are arcs in $\mathbb{R}^n$ which defy classification according to the developed criterion. We conclude with several conjectures and open questions.

Throughout, we will write $\{X; f_1, f_2, \ldots f_k\}$ where X is a complete metric space and $\{f_i\}$ are contraction maps on X to denote an iterated function system. The unique attractor $A \subset X$ associated with this IFS is invariant in the sense that $A = f_1(A) \bigcup f_2(A) \bigcup \ldots f_k(A)$.

2 Variation and Preliminaries

We shall let $I = [0,1]$ denote the unit interval from here forward. If $e : I \to \mathbb{R}^n$ is an embedding, the *variation on* e is defined as follows: Let $P = \{x_0 = 0 < x_1 < x_2 < \ldots < x_k = 1\}$ be a partition of I. Let $V_e(P) = \sum_{i=1}^{k} |e(x_i) - e(x_{i-1})|$ where $|a - b|$ denotes the usual Euclidean distance from a to b in $\mathbb{R}^n$. Then, $V_e = \sup\{V_e(P) : P \text{ is a partion of } I\}$ is called the *variation of e on I*. The image of e is called an arc. It should be pointed out that the variation of e on I has only to do with the image of I under e, that is on the arc itself as a point-set. Therefore, if e' is any other embedding of I so that $e(I) = e'(I)$, $V_e = V_{e'}$. Hence, we are justified in talking about the variation of an arc without considering a particular embedding of I. When dealing with an arc A in $\mathbb{R}^n$, it is meant to be understood that the arc arises from some embedding $e : I \to R^n$. We mention this embedding only if it is necessary or convenient and disregard it otherwise. We define $V(A)$ to be the variation of any particular embedding of I which gives rise to the arc A without ambiguity. Because of this arrangement, we may consider an arc A to be ordered naturally (in one of two manners). Moreover, we shall frequently make reference to the endpoints of an arc in the natural way. Thus, if A is an arc with endpoints $a < b$, we freely talk about a point c of A satisfying $a < c < b$. Notationally, we will write $V(A) = V_a^b$ to denote the variation of an arc A with endpoints $a, b \in \mathbb{R}^n$. (So, $V_a^b = V_b^a$.) Related to this fact, we will define $V_a^a = 0$ in a degenerate-type setting.

Elementary Properties involving Variation
The following several observations are easy to check: Let A be an arc in $\mathbb{R}^n$. Then,

- $V(A) > 0$
- If a, b are the endpoints of an arc and $a < c < b$, then $V_a^c + V_c^b = V_a^b$.
- If $V(A) < \infty$, the function $v : A \to [0, \infty)$ defined by $v(x) = V_a^x$ is strictly increasing.

Along the same lines, we state the following propositions whose proofs are omitted as well.

Proposition 2.1 *Let $A \subset \mathbb{R}^n$ be an ordered arc with endpoints $a < b$. Suppose $V_a^b = V(A) = \lambda < \infty$. Then $v : A \to [0, \infty)$ defined by $v(x) = V_a^x$ is continuous.*

If X, Y are metric spaces with metrics d_X and d_Y respectively, recall that a map $f : X \to Y$ is said to be a Lipschitz map provided there is a real k so that $d_Y(f(x), f(y)) \leq k \cdot d_X(x, y)$ for all $x, y \in X$. Here, k is called a Lipschitz constant for f. It follows readily that every Lipschitz map is necessarily continuous. (Hence, in this terminology, a contraction (contraction map) is a Lipschitz map with a Lipschitz constant which is strictly less than 1.)

Proposition 2.2 *Let $X = X_d$ be a metric space and $f_1, f_2, \ldots, f_n$ be Lipschitz maps from X to X having Lipschitz constants $k_1, k_2, \ldots, k_n$ respectively. Then $f = f_n \circ f_{n-1} \circ \ldots \circ f_1$ is Lipschitz with Lipschitz constant $k = k_n \cdot k_{n-1} \cdot \ldots \cdot k_1$.*

3 Arcs of Finite Variation are Attractors

The main point of this section is to verify that any arc with finite variation is an attractor of some IFS. To verify this, procedures for finding an IFS associated with a given arc are developed.

Theorem 3.1 *Let A be an arc in $\mathbb{R}^n$ and suppose $V(A) < \infty$. Then, A is the attractor of an IFS.*

Pf: For convenience, let $V(A) = \lambda$. Let $\sigma_i : \mathbb{R}^n \to \mathbb{R}$ denote projection onto the i^{th} coordinate for $i = 1, 2, \ldots, n$. Recall that σ_i is continuous and linear. Let $a < b$ denote the endpoints of A. Because $a \neq b$, there exists a j_0 such that $\sigma_{j_0}(a) \neq \sigma_{j_0}(b)$. Moreover, because A is compact, there exist $p, q \in A$ so that $\sigma_{j_0}(p) \leq \sigma_{j_0}(x) \leq \sigma_{j_0}(q)$ for every $x \in A$. As A is connected, $\sigma_{j_0}(A)$ is connected and if $\sigma_{j_0}(p) \leq t \leq \sigma_{j_0}(q)$, there exists $z \in A$ such that $\sigma_{j_0}(z) = t$.

Let J denote the interval of the j_0^{th} coordinate axis in R^n with endpoints $\sigma_{j_0}(p)$ and $\sigma_{j_0}(q)$. Let $r : \mathbb{R}^n \to J$ be the the composition of σ_{j_0} with the retraction of the the ${j_0}^{th}$ coordinate axis onto J. Then, r is a retraction of $\mathbb{R}^n$ onto J. As both maps of this composition are nonexpansive (i.e., have Lipschitz constant 1), the composition r is also nonexpansive. As the image of A under σ_{j_0} is J, observe that $r(A) = J$ too.

Because $V(A) = \lambda < \infty$, $v : A \to [0, \infty)$ defined by $v(x) = V_a^x$ is strictly increasing and, thus, invertible. The continuity of v together with the compactness and connectedness of A show that $v(A) = [0, \lambda]$ and so v is a closed map which implies that $v^{-1} : [0, \lambda] \to A$ is continuous too. That is v maps A homeomorphically onto $[0, \lambda]$.

Claim: v^{-1} is nonexpansive.

proof of claim: Let $x, y \in [0, \lambda]$ with $x < y$. Let $\alpha, \beta \in A$ denote the images of x and y under v^{-1} respectively; that is, $v^{-1}(x) = \alpha$ and $v^{-1}(y) = \beta$. As $x < y$ and v^{-1} is strictly increasing (because v is), $\alpha < \beta$ in the ordered arc A. Then $V_a^\alpha = x$ and $V_a^\beta = y$ and since $V_a^\beta = V_a^\alpha + V_\alpha^\beta$, $y = x + V_\alpha^\beta$. Hence, $y - x = V_\alpha^\beta \geq |\beta - \alpha| = |v^{-1}(x) - v^{-1}(y)|$. Rearranging, $|v^{-1}(x) - v^{-1}(y)| = |v^{-1}(y) - v^{-1}(x)| \leq y - x$. That is, v^{-1} is nonexpansive.

Choose Lipschitz maps $c_1, c_2, \ldots, c_k$ where $c_i : J \to [0, \lambda]$ with each with Lipschitz constant $\frac{1}{2}$ which cover $[0, \lambda]$. (Although the exact number of contraction maps is of no real significance, note that we could take k to be the the smallest positive integer greater than or equal to the reciprocal of $\frac{2\lambda}{|\sigma_j(q) - \sigma_j(p)|}$) to know that the number is finite.

Define $f_i = v^{-1} \circ c_i \circ r$. Thus $f_i : R^n \to A$ is Lipschitz map with Lipschitz constant no larger than $\frac{1}{2}$ (as v^{-1} and r are both nonexpansive). Then, f_i is a retraction for each i and it is easily checked that $A = \bigcup_{i=1}^{k} f_i(A)$. Hence, A is the attractor of the IFS $\{\mathbb{R}^2; f_1, f_2, \ldots, f_k\}$.

Corollary 3.1 *Let $A_1, A_2, A_3, \ldots, A_m$ be arcs in $\mathbb{R}^n$, each with finite variation. Then $\bigcup_{i=1}^{m} A_i$ is the attractor of an IFS in $\mathbb{R}^n$.*

Proof: The corollary rests on the fact that in the above proof, if S is any subset of $\mathbb{R}^n$, then $f_i(A \cup S) = f_i(A)$ for each of the contractions f_i constructed in the proof. Using contractions obtained from each arc A_i individually (say $\{f_{(i,1)}, f_{(i,2)}, \ldots, f_{(i,m_i)} : 1 \leq i \leq m\}$, the IFS formed from the total collection of contractions

$$\bigcup_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m_i}} f_{(i,j)} \text{ has } \bigcup_{i=1}^{m} A_i$$

as its attractor.

4 Identifying Arcs which are Not Attractors

To begin to establish the main idea of this work, namely, a criterion for identifying arcs which are not attractors of any IFS, we use the following proposition.

Proposition 4.1 *Let A be an arc in $\mathbb{R}^n$. Let C denote a subarc of A. Then, if $f : A \to A$ is Lipschitz with Lipschitz constant k, then $V(f(C)) \leq k \cdot V(C)$.*

Comment: While there are no restrictions on the map f such as requiring f to embed C in A, we note that $V(f(C))$ still makes sense: $f(C)$ is compact and connected in A and is thus is either a point or a subarc of A. The variation of either is defined.

Proof: If $V(C) = \infty$, the statement holds trivially. Suppose $V(A) < \infty$, whence $V(C) < \infty$. If f is constant on C, $f(C)$ is a point of A and

$V(f(C)) = 0$ by definition and so $V(f(C)) \leq k \cdot V(C)$ as claimed. Assume $f(C)$ is a subarc of A. Let P denote a partion $P = \{x_0 < x_1 < \ldots < x_t\}$ of $f(C)$. Construct a partion of C as follows: Let $c_0 < c_1$ denote the endpoints of C so that $f(c_0) \leq f(c_1)$ in $f(C)$ (using one of the natural two orders on C). Choose y_0 so that $c_0 \leq y_0$ with $f(y_0) = x_0$. Then choose y_1 so that $f(y_1) = x_1$ and so that if $z \in C$ with $f(z) = x_1$, then $y_1 \leq z$. That is, we are choosing y_i so that $f(y_i) = x_i$ and y_i is the smallest such point of C so that $y_i > y_{i-1}$ with this feature. Continue on in this manner to choose $y_2, y_3, \ldots y_t$. This construction gives rise to a partition $P' = \{c_0 \leq y_0 < y_1 < y_2 < \ldots < y_t \leq c_1\}$ of C. Then, by definition of variation and the fact f is Lipschitz, $|x_i - x_{i+1}| = |f(y_i) - f(y_{i+1})| \leq k \cdot |y_i - y_{i-1}|$ for each $i \in \{0, 1, \ldots, t-1\}$, and so, we have $\sum_{i=1}^{t} |x_i - x_{i-1}| \leq k \cdot \sum_{i=1}^{t} |y_i - y_i - 1|$. Since this inequality holds for every partition of $f(C)$, we have $V(f(C)) = \sup\left\{\sum_{i=1}^{t} |x_i - x_{i-1}|\right\} \leq k \cdot \sum_{i=1}^{t} |y_i - y_i - 1| \leq k \cdot \sup\left\{\sum_{i=1}^{t} |y_i - y_i - 1|\right\} = V(C)$ where the supremum on the left is taken over all partitions of $f(C)$ and the supremum on the right is taken over all partitions of C.

Proposition 4.2 *Let A be an arc in $\mathbb{R}^n$ with endpoints $a < b$. Suppose $\{c_m\}$ is a sequence of points of A satisfying*

1. $c_0 = a$

2. $c_m \to b$ *as* $m \to \infty$

Then, for $j \in \mathbb{N}$, if B_j denotes the possibly degenerate subarc of A with endpoints c_{j-1} and c_j, then, $V_a^b = V(A) \leq \sum_{j=1}^{\infty} V(B_j)$.

Proof: Let $P = \{x_0 = a < x_1 < x_2 < \ldots x_r = b\}$ be a partition of A. Let $\epsilon > 0$. Because $c_m \to b$, there exists $N \in \mathbb{N}$ so that $|c_N - b| < \epsilon$ and $c_N > x_{r-1}$. Let $P^* = P \bigcup \{c_0, c_1, \ldots, c_N\}$ be a refinement of P. Relabel $P^* = \{y_0 = a < y_1 < y_2 < \ldots y_{t-1} = c_N < y_t = b\}$. Let P_j^* denote the restriction of the partition P^* to the subarc B_j. Then, every term of $V(A, P^*)$ except for $|y_t - y_{t-1}|$ is added in the sum $\sum_{j=1}^{N} V(B_j, P_j^*)$. Hence, $V(A, P^*) - |y_t - y_{t-1}| \leq \sum_{j=1}^{N} V(B_j, P_j^*) \leq \sum_{j=1}^{N} V(B_j)$. As $|y_t - y_{t-1}| = |c_N - b| < \epsilon$, we have

$$V(A, P) \leq V(A, P^*) \leq \sum_{j=1}^{N} V(B_j, P_j^*) + \epsilon \leq \sum_{j=1}^{N} V(B_j) + \epsilon \leq \sum_{j=1}^{\infty} V(B_j) + \epsilon$$

Since this holds for any partition P, we have $V(A) \leq \sum_{j=1}^{\infty} V(B_j) + \epsilon$. As $\epsilon > 0$ is arbitrary, $V(A) \leq \sum_{j=1}^{\infty} V(B_j)$ as desired.

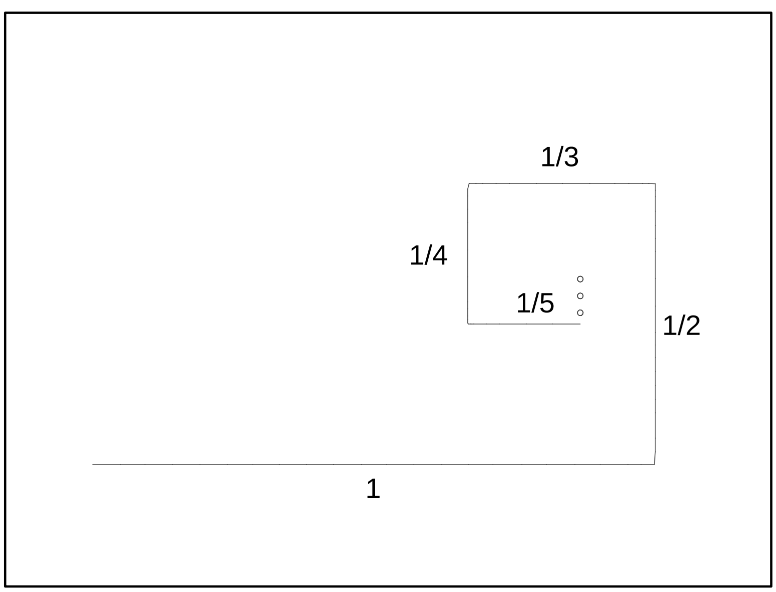
We are ready to present a theorem which provides many examples of nonattractors of IFSs. The theorem will be stated first, then an example of an arc in $\mathbb{R}^2$ meeting the hypotheses of the theorem will be given. The proof will be given after the example.

Theorem 4.1 *Let A be an arc in R^n with endpoints $a < b$. Suppose*

1. $V_x^y < \infty$ *for all* $x, y \in A$ *with* $x, y \neq b$*, and*
2. $V_x^b = \infty$ *for all* $x \neq b$*.*

Then, for any finite set of contractions $\{w_1, w_2, \dots w_N\}$ *on* R^n*,* A *is not the attractor of the IFS* $\{R^n; w_1, w_2, \dots, w_N\}$*.*

Example 1 The Harmonic Spiral*: The construction of this arc is based on the divergent harmonic series. We'll use the descriptive terms north, south, east, and west to label directions that correspond to directions in the plane in order to describe the arc: Start at the origin. Proceed 1 unit east. Turn north and proceed 1/2 unit. Turn west and proceed 1/3 unit. Turn south and proceed 1/4 unit. Turn east and proceed 1/5 unit. Continue on in this fashion. The desired arc spirals endlessly around a point that is related to the harmonic series. An illustration follows …*



Proof of Theorem 4.1: (By contradiction.) Assume that A is the attractor of some IFS $\{R^n; w_1, w_2, \ldots, w_N\}$. The first claim is that there exists a nonconstant contraction w_J so that $b \in w_J$. To see this, note that there exists a contraction w_J such that $b \in w_J(A)$ and therefore, w_J must be constant if the claim is false. So, if the claim is false, we may assume without loss of generality that $b \notin w_i(A)$ if $i \neq J$. But, then, A could be written as a union of two nonempty, disjoint closed sets, namely $\{b\} \bigcup_{i \neq J} w_i(A)$. This violates the fact that A is connected and establishes the claim. (Thus, in fact, we can assume that none of the contractions is the constant map.) Relabel w_J as f for convenience.

Now, the claim is that b must be a fixed point of f. Suppose to the contrary that $f(b) \neq b$. Then, there exists $t \in A$ so that $f(t) = b$. As f is a contraction on $\mathbb{R}^n$, f has a fixed point and as the image of f is contained in A, we can find $s \in A$ so that $f(s) = s$. Hence, by these assumptions, $s \neq b$ and $t \neq b$. Also, $f(s) = s \neq b$, but $f(t) = b$. By hypothesis then, $V_s^t < \infty$. Also by hypothesis, $V_{f(s) \neq b}^{f(t)=b} = \infty$. But, as $f(s), f(t)$ lie in the image of the arc with endpoints s and t, it follows from Proposition 4.1 and the definition of variation of an arc that $V_{f(s)}^{f(t)} < V_s^t < \infty$. This contradiction establishes the result that $f(b) = b$.

Let k denote the contractivity factor of f. The next claim is that $f^m(a) \neq b$ for any $m \in \mathbb{N}$. By definition, let $f^0(a) = a$. Note that $a \neq b$. Suppose $f(a) = b$. Then, because f is nonconstant, there exists $c \in A$ so that $f(c) \neq b$. Note that $c \neq b$. Then, by assumption $V_a^c < \infty$. But, $V_{f(a)=b}^{f(c) \neq b} = \infty$. This contradicts Proposition 4.1. Hence, $f(a) \neq b$. Inductively assume that for some positive integer k, $f^j(a) \neq b$ for every nonnegative integer $j \leq k$. If $f^{k+1}(a) = b$, then, $V_{f^k(a)}^{f^{k+1}(a)=b} = \infty$. But,$V_{f^{k-1}(a)}^{f^k(a)} < \infty$. This leads to a contradiction with Proposition 4.1 again. Hence, it must be that $f^m(a) \neq b$ for every $m \in \mathbb{N}$ as desired.

Recapping,

1. $f^m(a) \in A$ (as $f : A \to A$)
2. $f^m(a) \to b$ as $m \to \infty$ (as $|f^m(a) - b)| \leq k^m \cdot |a - b|$, using the contractivity of f applied m times and the fact $0 \leq k < 1$).)

Now for $j \in \mathbb{N}$, let B_j denote the, possibly degenerate, subarc of A with endpoints $f^{j-1}(a)$ and $f^j(a)$. Then, $V(B_1) = V_a^{f(a)} = L < \infty$ because B_1 has endpoints a and $f(a)$. Hence, $V(B_2) = V_{f(a)}^{f^2(a)} \leq k \cdot V(B_1) = k \cdot L$ by Proposition 4.1. Furthermore, for any positive integer j, we may inductively determine that $V(B_j) \leq k^{j-1} \cdot L$.Then, in Proposition 4.2, take the sequence

$\{c_m\}$ to be $\{f^m(a)\}$. The hypotheses of 4.2 are satisfied by 1,2, and 3 above so that $V(A) \leq \sum_{i=1}^{\infty} V(B_i)$. But,

$$\sum_{i=1}^{\infty} V(B_i) \leq L + kL + k^2L + \ldots = L \cdot \sum_{i=1}^{\infty} k^i$$

This last infinite series converges as $k < 1$. Hence, A is an arc of finite variation. This is a contradiction and it follows that A must not be the attractor of this IFS after all.

Upon the development of this criterion, several natural questions arise.

- What if the hypotheses of Theorem 4.1 are modified so that a single "bad" point occurs as an interior point of the arc (instead of the endpoint b in the theorem). Must the arc be a nonattractor too?
- Are arcs with a finite number of "bad" points nonattractors as well?
- Since arcs with infinite variation between each pair of distinct points (i.e., where all points are "bad") can be realized as an attractor of an IFS (see the Koch curve for instance), is it possible to construct an arc with this type of feature that must be an attractor of no IFS?

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