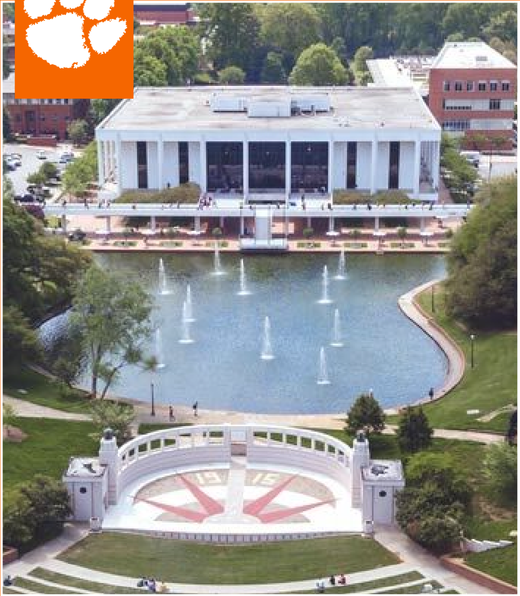


Code-based Cryptography: The Future of Security Against Quantum Threats

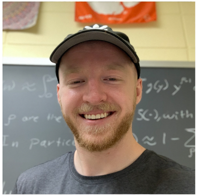
Felice Manganiello

Spring 2023 Section Meeting

April 29, 2023

Partially funded by
NSF DMS grant nr. 1547399

People Involved



Freeman Slaughter (Clemson University)

- Marco Baldi, Paolo Santini (Università Politecnica delle Marche)
- Alessandro Barenghi, Gerardo Pelosi (Politecnico di Milano)
- Sebastian Bitzer, Patrick Karl, Alessio Pavoni, Jonas Schupp, Antonia Wachter-Zeh, Violetta Weger (TUM)

Cryptography and Post-Quantum Cryptography

Coding Theory

Generic-Error Coding Theory

Zero-Knowledge Protocols

Cryptography

Cryptography is the practice and study of techniques for secure communication in the presence of adversarial behavior.

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- Authenticated pages (https)
- Digital signatures
- Zero-Knowledge protocols
- blockchain and cryptocurrencies
- etc.

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Number theory, commutative algebra, combinatorics, etc.

Cryptography Today

Problem **(Integer factorization - IF)**

Given a composite number N, find two integers a and b such that $ab = N$.

Easy: $N = 6$

Difficult: $N \approx 2^{2048} \approx 3.23 \cdot 10^{616}$

$\longrightarrow$ RSA cryptosystem (70's)

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Difficult: $N \approx 2^{2048} \approx 3.23 \cdot 10^{616}$

$\longrightarrow$ RSA cryptosystem (70's)

Problem (Discrete logarithm problem - DLP)

Given a cyclic group $G = \langle g \rangle$ and an element $a \in G$, find $e \in \mathbb{N}$ such that $a = g^e$.

Easy: $\mathbb{R}_{>0}$

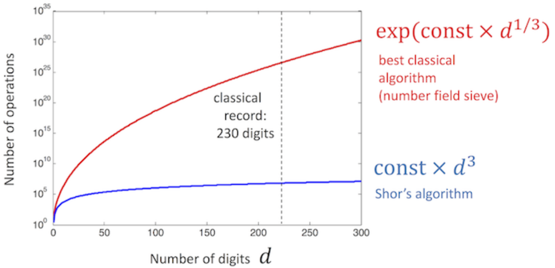
Difficult: $|G| \approx 2^{2048} \approx 3.23 \cdot 10^{616}$

$\longrightarrow$ Diffie-Hellman key exchange (70's)

Quantum computers and their threat

Theorem (Shor's Algorithm - '94)

There exists a polynomial-time quantum algorithm that breaks IF and DLP.



[1]

[1] image credit: https://quantum-computing.ibm.com/composer/docs/iqx/guide/shors-algorithm

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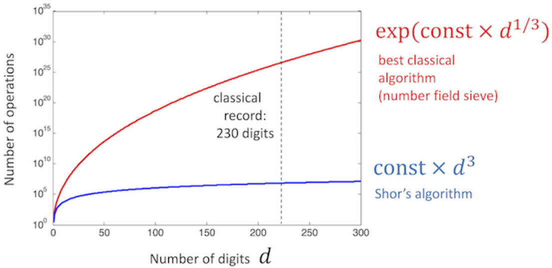

[1]

Remark

A full-scale quantum computer can break today's public key crypto!!

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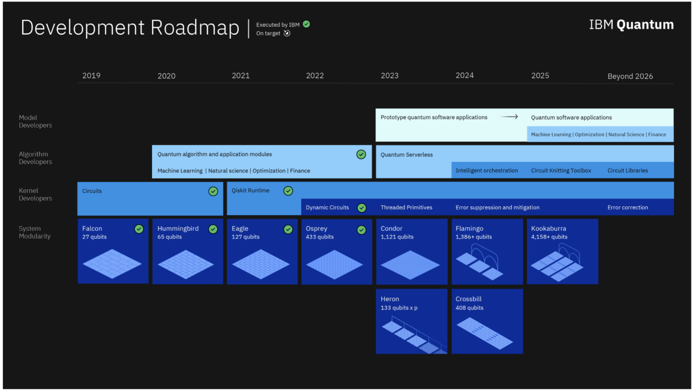
Progress in quantum computing



[2]

Remark

Some experts predict 10-15 years, no one knows for sure.

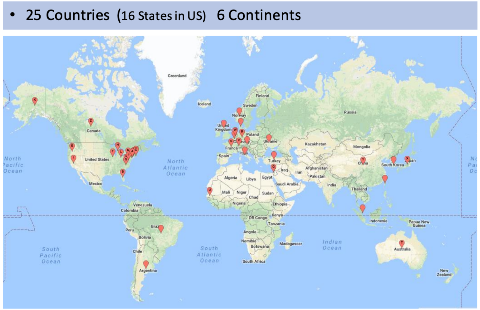
[2] image credit: https://www.ibm.com/quantum/roadmap

Post-quantum Cryptography and the NIST competition

Definition (Post-Quantum Cryptography (PQC))

Classical cryptographic algorithms which are secure against attacks by both classical and quantum computers.

- Dec 2, 2016: Call for proposal.
- Nov 30, 2017: Deadline
- 2018 - Round 1: 69 candidates
- 2019 - Round 2: 26 candidates
- 2020 - Round 3: 7 finalists and 8 alternates
- 2022 - NIST selects 4 finalists and 4 candidates



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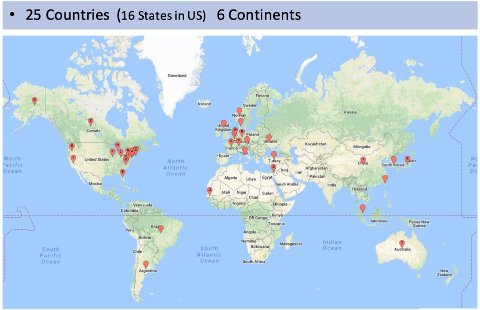
- NIST Call for Additional Digital Signatures



[a]

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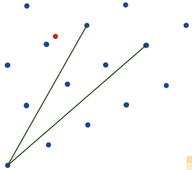
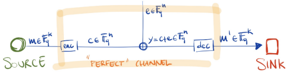
Goal: standards ready in about 1 year, complete compliance expected by 2035.

Post-Quantum Cryptography

Active research on:

- Lattice-based
- Code-based
- Multivariate
- Hash/Symmetric key-based signatures
- Isogeny-based





$$p^{(1)}(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(1)} \cdot x_i x_j \;+\; \sum_{i=1}^{n} p_i^{(1)} \cdot x_i + p_0^{(1)}$$

$$p^{(2)}(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=i}^{n} p_{ij}^{(2)} \cdot x_i x_j \;+\; \sum_{i=1}^{n} p_i^{(2)} \cdot x_i + p_0^{(2)}$$

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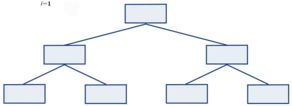

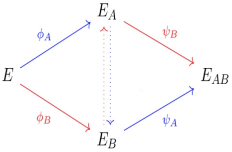

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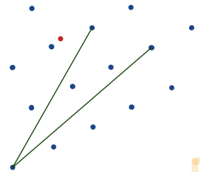

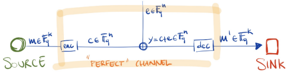

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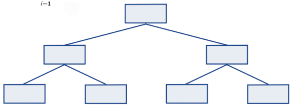

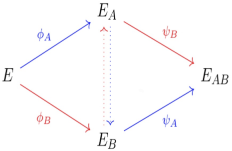

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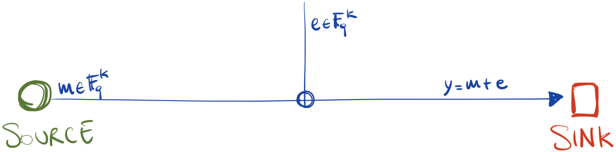
Noisy-Channel Coding Theorem - Shannon 1948)



Over $\mathbb{F}_2$:

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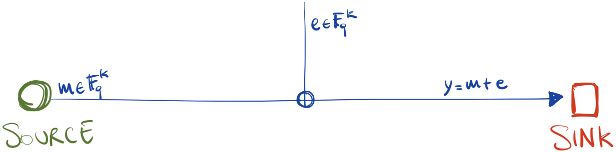

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Theorem (Noisy-Channel Coding Theorem - Shannon - 1948)

"In communication theory any channel, however affected by noise, possesses a specific channel capacity - a rate of conveying information that can never be exceeded without error, but that can, in principle, always be attained with an arbitrarily small probability of error."

Solved: Turbo codes (LTE networks), Polar & spatially-coupled LDPC codes (5G networks)

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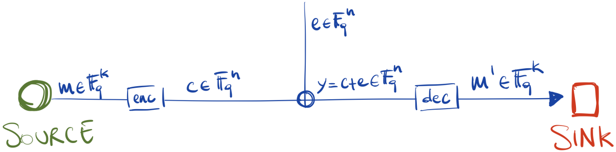

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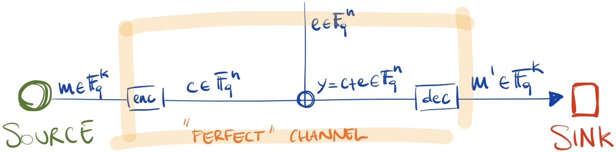

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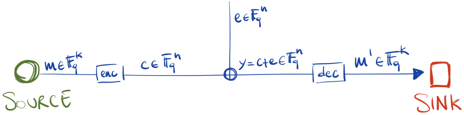
Error-Correcting codes



- $\mathbb{F}_q^k$ message space.

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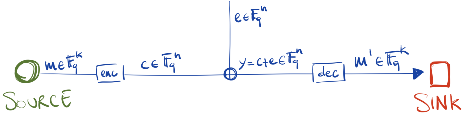

- $\mathbb{F}_q^k$ message space.
- $(\mathbb{F}_q^n, d_H)$ is a metric space with the Hamming distance

$$d_H(v, w) := wt(w - v) = |supp(w - v)| = \{i \in [n] \mid w_i \neq v_i\}.$$

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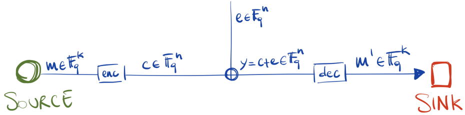

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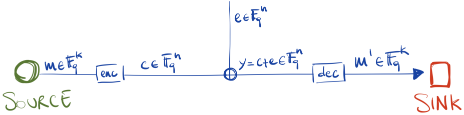

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- $\mathcal{C} :=$ enc$(\mathbb{F}_q^k) \subset \mathbb{F}_q^n$ is a $[n,k,d]_q$ linear code if it is a k-dimensional vector space and

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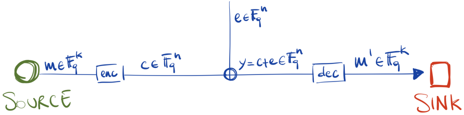

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- $\mathrm{enc} : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^n$ injective linear map.
- $\mathcal{C} := \mathrm{enc}(\mathbb{F}_q^k) \subset \mathbb{F}_q^n$ is a $[n,k,d]_q$ linear code if it is a k-dimensional vector space and

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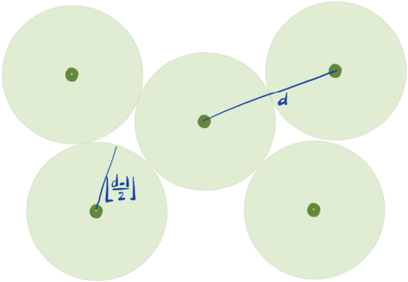
- $c = (c_1,\ldots,c_n) \in \mathcal{C}$ is a codeword.

Error-Correcting codes (cont'd)

Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a linear code with minimum distance d.



$$\begin{aligned} \pi : \mathbb{F}_q^n &\longrightarrow \mathcal{C} \\ y &\longmapsto \operatorname{argmin}\{d(y,c) \mid c \in \mathcal{C}\} \end{aligned}$$

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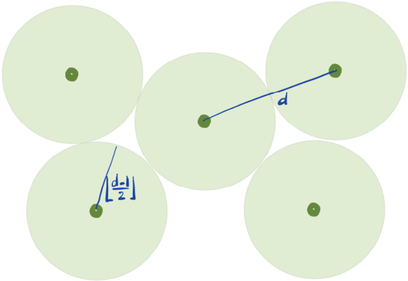

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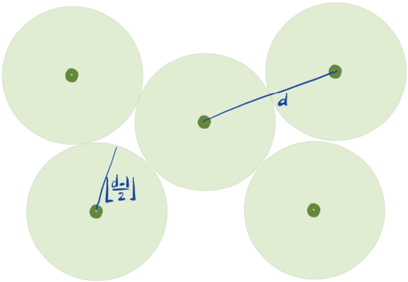

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dec is able to uniquely correct at least $\lfloor \frac{d-1}{2} \rfloor$ errors

Error-Correcting codes (cont'd)

Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a linear code.

- A generator matrix $G \in \mathbb{F}_q^{k \times n}$ for $\mathcal{C}$ is a fullrank matrix such that

$$\mathcal{C} = \mathrm{im}(G) = \{mG \mid m \in \mathbb{F}_q^k\}.$$

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- Syndrome of $y \in \mathbb{F}_q^n$ is $s_y := yH^t \in \mathbb{F}_q^{n-k}$.

Example: repetition code

- $\mathbb{F}_2$ message space
- $\mathsf{enc} : \mathbb{F}_2 \to \mathbb{F}_2^3$ such that

$$\mathsf{enc}(0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{enc}(1) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

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- $\mathcal{C}$ is a $[3, 1, 3]$ linear code that corrects 1 error.
- $G := \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and $H := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

The Syndrome Decoding Problem

Problem

For an $[n, k]$ code $\mathcal{C}$ with parity-check matrix $H \in \mathbb{F}_q^{(n-k)\times n}$, a syndrome $s \in \mathbb{F}_q^{n-k}$, and some $t \in \mathbb{N}$, find a vector $e \in \mathbb{F}_q^n$ such that $eH^t = s$ and $wt(e) = t$.

Theorem (Berlekamp et al. 1978, and Barg 1997)

This problem is NP-complete.

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Difference Sets

Let $\mathbb{F}_q$ be the field with q elements, with $q = p^N$ a prime power.

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For a k-element set $E \subseteq \mathbb{F}_q^n$, let $\langle E \rangle_{\mathbb{F}_p}$ be the span of E over $\mathbb{F}_p$:

$$\langle E \rangle_{\mathbb{F}_p} = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_k e_k \text{ for } \lambda_i \in \mathbb{F}_p, e_i \in E.$$

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For any set E, the set difference of E is

$$\Delta E = \{e_1 - e_2 \mid e_1, e_2 \in E\}.$$

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Theorem (M.,Slaugther 2023)

For a set $E \subseteq \mathbb{F}_q^n$, the chain $E \subseteq \Delta E \subseteq \Delta^2 E \subseteq \ldots$ stabilizes. That is, there exists some $k \in \mathbb{N}$ such that $\Delta^k E = \Delta^{k+1} E$. In this case, $\Delta^k E = \langle E \rangle_{\mathbb{F}_p}$.

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$$\Delta E = \{e_1 - e_2 \mid e_1, e_2 \in E\}.$$

Theorem (M.,Slaugther 2023)

For a set $E \subseteq \mathbb{F}_q^n$, the chain $E \subseteq \Delta E \subseteq \Delta^2 E \subseteq \ldots$ stabilizes. That is, there exists some $k \in \mathbb{N}$ such that $\Delta^k E = \Delta^{k+1} E$. In this case, $\Delta^k E = \langle E \rangle_{\mathbb{F}_p}$.

Definition

For a set E, the Δ-closure of E is $\overline{E}^{\Delta} = \lim_{k \to \infty} \Delta^k E$. We say that E is Δ-closed if $E = \overline{E}^{\Delta}$.

Generic Error Sets

Definition

An error set $E \subseteq \mathbb{F}_q^n$ is detectable by some code $\mathcal{C} \subseteq \mathbb{F}_q^n$ if $E \cap \mathcal{C} = \{0\}$. Similarly, this set of errors E is correctable by $\mathcal{C}$ if $\Delta E \cap \mathcal{C} = \{0\}$.

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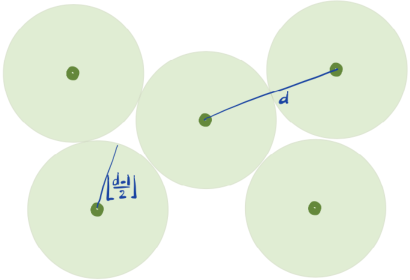
Example

In the case of Hamming balls, $\Delta B_t(0) \subseteq B_{d-1}(0)$, where $t = \lfloor \frac{d-1}{2} \rfloor$. This means that any error detectable under the set difference definition is also detectable under the minimum distance of a code.



Detection and Correction

It follows that Δ-closed sets are maximal sets for which detectability corresponds to correctability.

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Corollary

Given a code , a set E is detectable and correctable if and only if E is Δ-closed, meaning that $\overline{E}^{\Delta} = E$.

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Given a code , a set E is detectable and correctable if and only if E is Δ-closed, meaning that $\overline{E}^{\Delta} = E$.

Proposition

Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code with parity-check matrix $H \in \mathbb{F}_q^{(n-k)\times n}$. The set $E \subseteq \mathbb{F}_q^n$ is correctable by $\mathcal{C}$ if and only if its syndromes are unique, meaning that for $e, e' \in E$,

$$eH^t = e'H^t \iff e = e'.$$

Gilbert-Varshamov Bound

Theorem (M., Slaughter 2023)

There exists a code $\mathcal{C}$ correcting E once

$$|\Delta E| < q^{n-k}.$$

Gilbert-Varshamov Bound

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This recovers the standard Gilbert-Varshamov bound by taking $E \subseteq B_t(0)$:

Theorem (Gilbert-Varshamov Bound)

Let n, k, and d be such that

$$\sum_{i=1}^{d-1} \binom{n}{i} (q-1)^i < q^{n-k}.$$

Then there exists $\mathcal{C}$ an $[n, k]$ code $\mathcal{C}$ of minimum distance d.

GE-SDP

Problem (SDP)

For an $[n, k]$ code $\mathcal{C}$ with parity-check matrix $H \in \mathbb{F}_q^{(n-k)\times n}$, a syndrome $s \in \mathbb{F}_q^{n-k}$, and some $t \in \mathbb{N}$, find a vector $e \in \mathbb{F}_q^n$ such that $eH^t = s$ and $wt(e) = t$.

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Proposition (M., Slaughter 2023)

The GE-SDP is NP-complete.

Complexity of already known SPDs

SDP

$E = B_t(0)$

NP-complete[a]

[a] Berlekamp *et al.* 1978, and Barg 1997

Restricted SDP

$E = \{0, \pm 1\}^n$

NP-complete[a]

[a] Baldi *et al.* 2020

Rank SDP

$E = B_t^R(0)$

?

R-SDP(G)

$E = G^n$

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Theorem (M., Slaughter 2023)

Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code and $E \subseteq \mathbb{F}_q^n$ an error set such that $\overline{E}^\Delta \cap \mathcal{C} = \{0\}$. Then the GE-SDP can be solved in $\mathcal{O}(n^3)$.

If $E = \{0, \pm 1\}^n$, then $\overline{E}^\Delta = \mathbb{F}_p^n$. In this case,

$$\frac{k}{n} \leq \frac{N-1}{N},$$

meaning that the SDP might be easy for code with low rates.

Cryptography and Post-Quantum Cryptography

Coding Theory

Generic-Error Coding Theory

Zero-Knowledge Protocols

Zero-Knowledge Protocols (ZKP)

A ZKP is a method by which one party (the prover) can prove to another party (the verifier) that a given statement is true while the prover avoids conveying any additional information apart from the fact that the statement is indeed true.

A zero-knowledge proof must satisfy three properties:

- *Completeness*: an honest prover can convince a verifier.
- Soundness: a cheating prover can convince a verifier with a probability less than 1.
- Zero-Knowledge: the verifier learns nothing other than the statement's veracity.

Zero-knowledge protocol (ZKP) based on DLP

Private: $x \in \mathbb{N}$

Public: g such that $\mathbb{F}_p^* = \langle g \rangle$, and $y = g^x$

Prover

Verifier

Zero-knowledge protocol (ZKP) based on DLP

Private: $x \in \mathbb{N}$ — Public: g such that $\mathbb{F}_p^* = \langle g \rangle$, and $y = g^x$

Prover		**Verifier**
Create $x = x_1 + \cdots + x_n \pmod{p-1}$		
Compute $y_i = g^{x_i}$ for all i		
$(y_1, \ldots, y_n)$	$\longrightarrow$	

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	$\longleftarrow$	$j \in \{1, \ldots, n\}$
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		Checks $y_i = g^{x_i}$ for $i \neq j$ and $\prod_{i=1}^{n} y_i = y$

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$(y_1, \ldots, y_n)$ $\longrightarrow$

Verifier

$\longleftarrow$ $j \in \{1, \ldots, n\}$

$(x_1, \ldots, x_n)$ except for j $\longrightarrow$

Checks $y_i = g^{x_i}$ for $i \neq j$ and $\prod_{i=1}^{n} y_i = y$

Figure: Prover - Jess

Figure: Verifier - Felice

GE-CVE - a ZKP based on GE-SDP (M., Slaughter 2023)

[3]

Public data: $q, n, k \in \mathbb{N}, E \subset \mathbb{F}_q^n, H \in \mathbb{F}_q^{(n-k)\times n}$

Private Key: $e \in E$

Public Key: $s = eH^t \in \mathbb{F}_q^{n-k}$

PROVER		VERIFIER
$u \xleftarrow{\$} \mathbb{F}_q^n,\ M \xleftarrow{\$} \mathfrak{S}_E$		
Set $c_0 = \text{Hash}(M, uH^t)$		
Set $c_1 = \text{Hash}(uM, eM)$	$\xrightarrow{(c_0, c_1)}$	
	$\xleftarrow{z}$	$z \xleftarrow{\$} \mathbb{F}_q^*$
Set $y = (u + ze)M$	$\xrightarrow{y}$	
	$\xleftarrow{b}$	Choose $b \in \{0, 1\}$
If $b = 0,$ set $f := M$		
If $b = 1,$ set $f := eM$	$\xrightarrow{f}$	
		If $b = 0,$ accept if $c_0 = \text{Hash}(f, (yf^{-1})H^t - zs)$. If $b = 1,$ accept if $f \in E$ and $c_1 = \text{Hash}(y - zf, f)$.

[3] Adaptation of CVE by Cayrel, Veron, El Yousfi Alaoui 2010

GE-CVE - a ZKP based on GE-SDP (M., Slaughter 2023)

This is genuinely a zero-knowledge identification scheme:

- Completeness: an honest prover can convince a verifier.
- Soundness: a cheating prover can convince a verifier with only a small probability $\left(\frac{q}{2(q-1)}\right)$.
- Zero-Knowledge: the verifier learns nothing other than the statement's veracity.

Future for GE-SDP

We plan on submitting on June 1, 2023 a digital signature scheme based on R-SDP(G)

Universities involved:

- Clemson University
- Università Politecnica delle Marche
- Politecnico di Milano
- Technical University of Munich

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Thank you.