A College Algebra Proof
of the Fundamental Theorem of Algebra (FTA)

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Abstract

William Kingdon Clifford's FTA proof (i.e. a real polynomial factors into linear and quadratic terms) appearing only as an abstract has been overlooked among all FTA proofs as the only one accessible to college algebra students. We provide a manuscript for his abstract showing the quadratic factorization of even polynomials in quartic, sextic, octic, and decic cases; so, by extension any even degree polynomial. All that is needed from college algebra is 1. simple determinant evaluation and manipulation used to 2. eliminate a variable, say x, from two polynomial equations that leads directly to "Professor Sylvester's Dialytic Method's" determinant, 3. quadratic synthetic division, and 4. the intermediate value theorem. Forget geometry, inequalities, square roots, imaginaries, calculus, discriminants, symmetric functions, greatest common divisors, splitting fields, etc.

1 Clifford's Abstract [3]

PROOF THAT EVERY RATIONAL EQUATION HAS A ROOT *

(*Abstract*)

The proof contained in the present communication depends on the determination of a quadratic factor of the rational integral expression

$$x^{2s} + a_1x^{2s-1} + a_2x^{2s-2} + \cdots + a_{2s}$$

On dividing this expression by $x^2 + p_1x + p_2$, we obtain by the ordinary rules a remainder of the form $M_{2s-1}x + N_{2s}$ where M_{2s-1} and N_{2s} are functions of p_1 and p_2 whose weights are $2s - 1$ and $2s$ respectively, and which may accordingly be written in the forms

$$\begin{aligned} M_{2s-1} &= b_{2s-1} + p_2b_{2s-3} + \cdots\cdots + p_2^{s-1}b_1, \\ N_{2s} &= c_{2s} + p_2c_{2s-2} + \cdots\cdots + p_2^{s}, \end{aligned}$$

where the b, c are an order in p_1 indicated by their suffixes. On writing down (by Professor Sylvester's Dialytic method) the result of eliminating p_2 between these equations, it is at once apparent that the resultant is of the order $s(2s - 1)$ in p_1. Thus the determination of a quadratic factor of an expression is reduced to the solution of an equation of order $s(2s - 1)$. But this degree is *one more degree odd* than the original number $2s$; that is to say, if the degree $2s$ is 2^k multiplied by an odd number, then $s(2s - 1)$ is 2^{k-1} multiplied by an odd number. Hence by a repetition of this process we shall ultimately arrive at an equation of odd order, which, as is well known, must have a real root. By then retracing our steps the existence of a quadratic factor of the original expression is demonstrated.

* [From *Cambridge Philosophical Society Proceedings*, II. 1876. Read Feb. 21, 1870,pp.156,157.]

2 Introduction

Today there are more than 200 proofs of the FTA. Piotr Blaszczyk [2] cites two bibliographies, one [10] with nearly 100 proofs up to 1907 and a second [9] with 97 between 1933 and 2009. What about the gap between 1907 and 1933? Wikipedia FTA cites many proofs, some beyond 2009. Even though Clifford's proof is cited in [10], no FTA paper cites it or recognizes its elementary mathematics; so the belief that the proof of the FTA requires graduate mathematics is perpetuated.

When Clifford uses the term *resultant*, it means Sylvester determinant which evaluates as a polynomial. He doesn't exhibit one but according to [3] he acknowledged a similar FTA proof [8] that does. His sentence *On writing down (by Professor Sylvester's Dialytic method)* the result of *eliminating* p_2 between these equations, it is at once apparent that the *resultant* is of the order $s(2s-1)$ in p_1 implies Clifford is saying "trust my spade work using these concepts; otherwise, since I've given you their names, go verify them for yourself". That is what this paper does and more; namely, also eliminating p_1 between these equations, it is at once apparent that the *resultant* is of the order $s(2s - 1)$ in p_2. The two eliminations are not equivalent. Clifford's can be less than $s(2s - 1)$ when $p_1 = 0$ while p_2 can't equal 0 and be a quadratic. Thus the determination of a quadratic factor of an expression is reduced to the solution of an equation of order $s(2s - 1)$.

Clifford's observation of increasing polynomial degree to arrive at an odd degree also appears in Gauss's (1815) second of four proofs [6] and [11], praised as ingenious and entirely algebraic. We find it incomplete and unclear. It's a mystery how to interpret Gauss's treatment of polynomial discriminants, symmetric functions, and the greatest common denominator of an even polynomial and its derivative to find the number of factors, 6 and 15 for the simplest case of a quartic. Somehow the discriminant of a quartic leads to sixth degree polynomial which in turn's discriminant leads to a fifteenth degree polynomial whose factors all have non-zero discriminants. Contrast this to Clifford's most difficult concept of a Sylvester determinant using properties of determinants from a text such as Smith, K., Boyle, P. "College Algebra", ed. 4, Brooks/Cole, Pacific Grove, CA, (1989) 340-351 where coincidently, Clifford is featured on page 58 in a historical note. To aid in understanding our paper, first master the simplest case, a quartic. Gauss is credited for the term FTA.

3 Eliminating x from Two Polynomial Equations yields a Sylvester Determinant [4], [7]

3.1 Elimination of x from two linear equations

$$\begin{array}{cccll} a_0x & + & a_1 & =0 & (1) \\ b_0x & + & b_1 & =0 & (2) \end{array}$$

If the same value of x satisfies (1) and (2), then substituting x from (2) into (1) gives

$$a_0(-b_1/b_0) + a_1 = 0 \text{ or equivalently}$$

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} = 0$$

Thus the two equations have a common root if the determinant is zero, and conversely [4]. The determinant is known as a Sylvester determinant and its zero condition is the *elimination* of x (via subtraction rather than substitution):

Multiply (2) by a_0/b_0 and subtract it from (1) getting

$$a_1 - a_0b_1/b_0 = 0 \text{ or equivalently}$$

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} = 0$$

3.2 Elimination of x from a quadratic equation and a linear equation

$$\begin{array}{cccccll} a_0x^2 & + & a_1x & + & a_2 & =0 & (3) \\ & & b_0x & + & b_1 & =0 & (4) \end{array}$$

Multiply (4) by a_0/b_0x and subtract it from (3) getting

$$(a_1 - a_0b_1/b_0)x + a_2 = 0$$

which when combined with (4) is two linear equations:

$$\begin{array}{rccl} (a_1 - a_0b_1/b_0)x & + & a_2 & =0 \\ b_0x & + & b_1 & =0 \end{array}$$

Apply Sylvester's determinant for two linear equations to get

$$\begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 \\ b_0 & b_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & 0 \\ 0 & b_0 & b_1 \end{vmatrix} = \begin{vmatrix} 0 & a_1 - a_0b_1/b_0 & a_2 \\ b_0 & b_1 & 0 \\ 0 & b_0 & b_1 \end{vmatrix} = b_0 \begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 \\ b_0 & b_1 \end{vmatrix}$$

The Sylvester determinant for this case is the first one of three above. The second comes from the first by the determinant rule *multiplying a row by a constant and adding or subtracting it to another row leaves the determinant unchanged.* The third determinant comes from the second by *determinant evaluation.* It's the same as the top determinant because b_0 is not zero. This proves the Sylvester determinant is zero. This explanation applies to subsequent eliminations.

Because this case was reduced to the two linear equations case, it means that the quadratic equation and linear equation have a common root if and only if its Sylvester determinant is zero thereby propagating the common root property of zero Sylvester determinants to subsequent cases.

3.3 Elimination of x from two quadratic equations

$$\begin{array}{ccccccl} a_0x^2 & + & a_1x & + & a_2 & = 0 & (5) \\ b_0x^2 & + & b_1x & + & b_2 & = 0 & (6) \end{array}$$

Multiply (6) by a_0/b_0 and subtract it from (5) getting

$$(a_1 - a_0b_1/b_0)x + a_2 - a_0b_2/b_0 = 0$$

which when combined with (6) is a quadratic equation and a linear equation:

$$\begin{array}{cccccc} b_0x^2 & + & b_1x & + & b_2 & = 0 \\ & & (a_1 - a_0b_1/b_0)x & + & a_2 - a_0b_2/b_0 & = 0 \end{array}$$

Apply Sylvester's determinant for a quadratic equation and a linear equation to get

$$\begin{vmatrix} b_0 & b_1 & b_2 \\ a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & 0 \\ 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} = \begin{vmatrix} 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & 0 \\ 0 & 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} = b_0 \begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & 0 \\ 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 \\ b_0 & b_1 & b_2 \end{vmatrix}$$

3.4 Elimination of x from a cubic equation and a linear equation

$$\begin{array}{cccccccll} a_0x^3 & + & a_1x^2 & + & a_2x & + & a_3 & = 0 & (7) \\ & & & & b_0x & + & b_1 & = 0 & (8) \end{array}$$

Multiply (8) by a_0/b_0x^2 and subtract it from (7) getting

$$(a_1 - a_0b_1/b_0)x^2 + a_2x + a_3 = 0$$

which when combined with (8) is a quadratic equation and a linear equation:

$$\begin{array}{cccccc} (a_1 - a_0b_1/b_0)x^2 & + & a_2x & + & a_3 & = 0 \\ & & b_0x & + & b_1 & = 0 \end{array}$$

Apply Sylvester's determinant for a quadratic equation and a linear equation to get

$$\begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 & a_3 \\ b_0 & b_1 & 0 \\ 0 & b_0 & b_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & 0 & 0 \\ 0 & b_0 & b_1 & 0 \\ 0 & 0 & b_0 & b_1 \end{vmatrix} = \begin{vmatrix} 0 & a_1 - a_0b_1/b_0 & a_2 & a_3 \\ b_0 & b_1 & 0 & 0 \\ 0 & b_0 & b_1 & 0 \\ 0 & 0 & b_0 & b_1 \end{vmatrix} = b_0 \begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 & a_3 \\ b_1 & b_0 & 0 \\ 0 & b_0 & b_1 \end{vmatrix}$$

3.5 Elimination of x from a cubic equation and a quadratic equation

$$\begin{array}{cccccccll} a_0x^3 & + & a_1x^2 & + & a_2x & + & a_3 & = 0 & (9) \\ & & b_0x^2 & + & b_1x & + & b_2 & = 0 & (10) \end{array}$$

Multiply (10) by a_0/b_0x and subtract it from (9) getting

$$(a_1 - a_0b_1/b_0)x^2 + (a_2 - a_0b_2/b_0)x + a_3 - a_0b_3/b_0 = 0$$

which when combined with (10) is two quadratic equations:

$$\begin{array}{cccccc} (a_1 - a_0b_1/b_0)x^2 & + & (a_2 - a_0b_2/b_0)x & + & a_3 & = 0 \\ b_0x^2 & + & b_1x & + & b_2 & = 0 \end{array}$$

Apply Sylvester's determinant for two quadratic equations to get

$$\begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 & 0 \\ 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \end{vmatrix} = \begin{vmatrix} 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 & 0 \\ 0 & 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 \\ b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \end{vmatrix} =$$

$$b_0 \begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 & 0 \\ 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix}$$

3.6 Elimination of x from two cubic equations

$$\begin{array}{lllllllll} a_0x^3 & + & a_1x^2 & + & a_2x & + & a_3 & = 0 & (11) \\ b_0x^3 & + & b_2x^2 & + & b_1x & + & b_0 & = 0 & (12) \end{array}$$

Multiply (12) by a_0/b_0 and subtract it from (11) getting

$$(a_1 - a_0b_1/b_0)x^2 + (a_2 - a_0b_2/b_0)x + a_3 - a_0b_3/b_0 = 0$$

which when combined with (12) is a cubic equation and a quadratic equation:

$$\begin{array}{rcrcrcrl} b_0x^3 & + & b_1x^2 & + & b_2x & + & b_3 & = 0 \\ & & (a_1 - a_0b_1/b_0)x^2 & + & (a_2 - a_0b_2/b_0)x & + & a_3 - a_0b_3/b_0 & = 0 \end{array}$$

Apply Sylvester's determinant for a cubic equation and a quadratic equation getting

$$\begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 & 0 & 0 \\ 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 & 0 \\ 0 & 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 & 0 & 0 \\ 0 & 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 & 0 \\ 0 & 0 & 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} =$$

$$b_0 \begin{vmatrix} a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 & 0 & 0 \\ 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 & 0 \\ 0 & 0 & a_1 - a_0b_1/b_0 & a_2 - a_0b_2/b_0 & a_3 - a_0b_3/b_0 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix}$$

4 Definition of Sylvester Determinant

$$\begin{array}{rcl} p(x) & = & a_0x^m + a_1x^{m-1} + \cdots + a_m \\ q(x) & = & b_0x^n + b_1x^{n-1} + \cdots + b_n \end{array}$$

The pattern from the examples shows that a Sylvester determinant has the number of shifted rows of the coefficients of the polynomial of minimum degree is the degree of the polynomial of maximum degree plus the number of shifted rows of the coefficients of the polynomial of maximum degree is the degree of the polynomial of minimum degree. Equivalently, a Sylvester's determinant has $m+n$ rows where $\min(m,n)$ is the number rows of shifted coefficients of the polynomial of degree $\max(m,n)$ and the remaining rows are the shifted coefficients of the other polynomial. As stated above, a zero Sylvester determinant is the condition for a common root and is said to be the *elimination* of x from the two equations.

5 Quadratic Synthetic Division

We synthetically divide [1] a quartic, sextic, octic, and decic by the quadratic x^2-ax-b getting remainder $M(a,b)x+N(a,b)=0$. Our $-a$ and $-b$ are Clifford's p_1 and p_2. Like Clifford, we eliminate b getting a determinant in a at each step but also eliminate a getting a determinant in b at each step. When studying expressions and equations, make sure dimensions are consistent. For example, in our quadratic divisor x^2-ax-b, a has dimension 1 and b dimension 2 and the coefficients 1,A,B,C,D of our quartic have dimensions 0,1,2,3,4, respectively. Note that the determinant in b, is treated as if b is a single variable of degree 1 but of dimension 2. Thus in expressions involving b, b's dimension must be 2. For example, in the expression below for the elimination of b in the quartic case: $p(b)=(2a+A)b+a^3+Aa^2+Ba+C$ each term has dimension 3 and $q(b)=b^2+(a^2+Aa+B)b+D$ has dimension 4.

The following tableaux below show *quadratic* synthetic division for quartic, sextic, octic, and decic expressions.

5.1 Quartic: $(x^4+Ax^3+Bx^2+Cx+D)\div(x^2-ax-b), D>0$

	x^4	x^3	x^2	x^1	x^0
	1	A	B	C	D
a		$aT1$	$aT2$	$aT3$	$aT4=0$
b			$bT1$	$bT2$	$bT3$
total	1	$A+aT1$	$B+aT2+bT1$	$C+aT3+bT2=0$	$D+aT4+bT3=0$
	$=T1$	$=T2$	$=T3$	$=T4$	$=T5$

If quadratic synthetic division seems puzzling, consider the following identity:

$$(T1x^2+T2x^1+T3x^0+T4x^{-1}+T5x^{-2})(x^2-ax-b)=$$

$$\begin{array}{ccccccccc}
T1x^4 & +T2x^3 & +T3x^2 & +T4x^1 & +T5x^0 & & & & \\
 & -aT1x^3 & -aT2x^2 & -aT3x^1 & -aT4x^0 & & -aT5x^{-1} & & \\
 & & -bT1x^2 & -bT2x^1 & -bT3x^0 & & -bT4x^{-1} & -bT5x^{-2} & = \\
\hline
x^4 & +Ax^3 & +Bx^2 & +Cx^1 & Dx^0 & -(aT5+bT4)x^{-1} & & -bT5x^{-2} &
\end{array}$$

The last line shows that the quadratic (x^2-ax-b) is a factor of the quartic if and only if T4=T5=0 verifying the tableau quadratic synthetic division calculations. A similar result follows for the sextic, octic, and decic cases.

5.2 Sextic: $(x^6+Ax^5+Bx^4+Cx^3+Dx^2+Ex+F)\div(x^2-ax-b), F>0$

The tableau below only gives the last two columns as its first five are exactly those of a quartic multiplied by x^2 divided by a quadratic; hence, one only needs to prefix the tableau below with the quartic tableau adjusted so its powers run from x^6 to x^2 and none of quartic total terms are set to zero.

	x^1	x^0
	E	F
a	$aT5$	$aT6=0$
b	$bT4$	$bT5$
total	$E+aT5+bT4=0$	$F+aT6+bT5=0$
	$=T6$	$=T7$

5.3 Octic: $(x^8+Ax^7+Bx^6+Cx^5+Dx^4+Ex^3+Fx^2+Gx+H)\div(x^2-ax-b), H>0$

The tableau below give only the last two columns as its first seven are exactly those of a sextic multiplied by x^2 divided by a quadratic; hence, one only needs to prefix the tableau with the full sextic tableau adjusted so its powers run from x^8 to x^2 and none of the sextic total terms are set to zero.

	x^1	x^0
	G	H
a	$aT7$	$aT8=0$
b	$bT6$	$bT7$
total	$G+aT7+bT6=0$	$H+aT8+bT7=0$
	$=T8$	$=T9$

5.4 Decic: $(x^{10}+Ax^9+Bx^8+Cx^7+Dx^6+Ex^5+Fx^4+Gx^3+Hx^2+Ix+J)$ $\div(x^2-ax-b), J>0$

The tableau below give only the last two columns as its first nine are exactly those of an octic multiplied by x^2 divided by a quadratic; hence, one only needs to prefix the tableau with the full octic tableau adjusted so its powers run from x^{10} to x^2 and none of the octic total terms are set to zero.

	x^1	x^0
	I	J
a	$aT9$	$aT10=0$
b	$bT8$	$bT9$
total	$I+aT9+bT8=0$	$J+aT10+bT9=0$
	$=T10$	$=T11$

6 Sylvester Determinants

The Sylvester determinants below were computed from open source software wxMaxima, a Graphic User Interface to Maxima 5.44.0 derived from Massachusetts Institute Technology's computer algebra system Macsyma. Specifically, we applied wxMaxima functions: *matrix, determinant, expand, ratsimp, factor, solve.* The degree of a Sylvester determinant can be reduced from $m+n$ to $\max(m,n)$ using the row rule for determinants. This was done in the octic and decic cases of eliminating a because wxMaxima became overtaxed when the degree of a exceeded 9.

6.1 Quartic Determinants

6.1.1 Elimination of b

$$\begin{aligned} p(b) = &\quad T4=(2a+A)b+a^3+Aa^2+Ba+C=0 \quad (m=1)\\ q(b) = &\quad D+bT3=b^2+(a^2+Aa+B)b+D=0 \quad (n=2) \end{aligned}$$

$$\begin{matrix} b^2 & b^1 & b^0 \end{matrix}$$

$$\begin{vmatrix} 1 & a^2+Aa+B & D \\ 2a+A & a^3+Aa^2+Ba+C & 0 \\ 0 & 2a+A & a^3+Aa^2+Ba+C \end{vmatrix} =$$

$$a^6+3Aa^5+(3A^2+2B)a^4+(A^3+4AB)a^3+(2A^2B+AC+B^2-4D)a^2+A(AC+B^2-4D)a+A(BC-AD)-C^2=0$$

6.1.2 Elimination of a

$$\begin{aligned} p(a) = &\quad T4=a^3+Aa^2+a(2b+B)+Ab+C=0 \quad (m=3)\\ q(a) = &\quad D+bT3=ba^2+Aba+b^2+Bb+D=0 \quad (n=2) \end{aligned}$$

$$\begin{matrix} a^4 & a^3 & a^2 & a^1 & a^0 \end{matrix}$$

$$\begin{vmatrix} 1 & A & 2b+B & Ab+C & 0 \\ 0 & 1 & A & 2b+B & Ab+C \\ b & Ab & b^2+Bb+D & 0 & 0 \\ 0 & b & Ab & b^2+Bb+D & 0 \\ 0 & 0 & b & Ab & b^2+Bb+D \end{vmatrix} =$$

$$b^6+Bb^5+(AC-D)b^4+((A^2-2B)D+C^2)b^3+(AC-D)Db^2+BD^2b+D^3=0$$

6.2 Sextic Determinants

6.2.1 Elimination of b

$$\begin{aligned} p(b) = &\quad T6=(3a+A)b^2+(4a^3+3Aa^2+2Ba+C)b+a^5+Aa^4+Ba^3+Ca^2+Da+E=0 \quad (m=2)\\ q(b) = &\quad F+bT5=b^3+(3a^2+2Aa+B)b^2+(a^4+Aa^3+Ba^2+Ca+D)b+F=0 \quad (n=3) \end{aligned}$$

$$
\begin{array}{lllll}
b^4 & b^3 & b^2 & b^1 & b^0 \\
\end{array}
$$
$$
\left|\begin{array}{lllll}
1 & 3a^2+2Aa+B & a^4+Aa^3+Ba^2+Ca+D & F & 0 \\
0 & 1 & 3a^2+2Aa+B & a^4+Aa^3+Ba^2+Ca+D & F \\
3a+A & 4a^3+3Aa^2+2Ba+C & a^5+Aa^4+Ba^3+Ca^2+Da+E & 0 & 0 \\
0 & 3a+A & 4a^3+3Aa^2+2Ba+C & a^5+Aa^4+Ba^3+Ca^2+Da+E & 0 \\
0 & 0 & 3a+A & 4a^3+3Aa^2+2Ba+C & a^5+Aa^4+Ba^3+Ca^2+Da+E
\end{array}\right| =
$$

$$
\begin{gathered}
a^{15} + 5a^{14} + (10A^2 + 4B)a^{13} + (10A^3 + 16AB + 2C)a^{12} + \\
(5A^4 + 24A^2B + 9AC - 2D)a^{11} + (A^5 + 16A^3B + 18AB^2 + (15A^2 + 6B)C - 4AD - 10E)a^{10} + \cdots
\end{gathered}
$$

$$
\begin{gathered}
+(-27AF^2 + (-4AB^2 - 9A^2BC + 3AC^2 + (7A^3 + 18AB)D^2 + 21AE^2)F - (4A^3 + 9AB - 6C)E^2 + \\
((AB^2C + A^2C^2 + 3(A^2B - C)E - 4AD^3 - (AB^2 - A^2C)D^2 - 2(ABC^2 - C^3))a^2 + \\
(-A^2F^2 + (4A^3 + 6AB - 9C)E + (4A^2B + 3AC)D - (A^2 - 3B)C^2 - 4AB^2C)F + \\
(-4AB^2 - 4AC + B^2 - 3D)E^2 + (-ABC^2 + 2AD^2 + C^3 - BCD)E + A^2D^3 + (-ABC + C^2)D^2)a - \\
A^3F^2 + ((2A^2B - 3AC)E + A^2CD - ABC^2 + C^3)F - E^3 + (-AB^2 + 2AD + BC)E^2 - (A^2D^2 + (C^2 - ABC)D)E = 0
\end{gathered}
$$

6.2.2 Elimination of a

$$
\begin{array}{rlr}
p(a) = & T6 = a^5 + Aa^4 + (4b + B)a^3 + (3Ab + C)a^2 + (3b^2 + 2Bb + D)a + Ab^2 + Cb + E = 0 & (m = 5) \\
q(a) = & F + bT5 = ba^4 + Aba^3 + (3b^2 + Bb)a^2 + (2Ab^2 + Cb)a + b^3 + Bb^2 + Db + F = 0 & (n = 4)
\end{array}
$$

$$
\begin{array}{lllllllll}
a^8 & a^7 & a^6 & a^5 & a^4 & a^3 & a^2 & a^1 & a^0
\end{array}
$$
$$
\left|\begin{array}{lllllllll}
1 & A & 4b+B & 3Ab+C & 3b^2+2Bb+D & Ab^2+Cb+E & 0 & 0 & 0 \\
0 & 1 & A & 4b+B & 3Ab+C & 3b^2+2Bb+D & Ab^2+Cb+E & 0 & 0 \\
0 & 0 & 1 & A & 4b+B & 3Ab+C & 3b^2+2Bb+D & Ab^2+Cb+E & 0 \\
0 & 0 & 0 & 1 & A & 4b+B & 3Ab+C & 3b^2+2Bb+D & Ab^2+Cb+E \\
b & Ab & 3b^2+Bb & 2Ab^2+Cb & b^3+Bb^2+Db+F & 0 & 0 & 0 & 0 \\
0 & b & Ab & 3b^2+Bb & 2Ab^2+Cb & b^3+Bb^2+Db+F & 0 & 0 & 0 \\
0 & 0 & b & Ab & 3b^2+Bb & 2Ab^2+Cb & b^3+Bb^2+Db+F & 0 & 0 \\
0 & 0 & 0 & b & Ab & 3b^2+Bb & 2Ab^2+Cb & b^3+Bb^2+Db+F & 0 \\
0 & 0 & 0 & 0 & b & Ab & 3b^2+Bb & 2Ab^2+Cb & b^3+Bb^2+Db+F
\end{array}\right| =
$$

$$
\begin{gathered}
b^{15} + Bb^{14} + (AC - D)b^{13} + (F - AE + (A^2 - 2B)D + C^2)b^{12} + \\
((2B - A^2)F + (C - 3AB + A^3)E - D^2 + ACD)b^{11} + \cdots
\end{gathered}
$$

$$
\begin{gathered}
+(2(A^2 - B)F^3 + ((3C - 2AB)E + 2D^2 + (B^2 - 2AC)D)F^2 + (AC - 4D)E^2F + E^4)b^5 + \\
((2D + AC - B^2)F^3 + ((BC - 3AD)E - E^2)F^2 + AE^3F)b^4 + \\
(F^4 - (AE + 2BD - C^3)F^3 + BE^2F^2)b^3 + (CEF^3 - BF^4)b^2 + DF^4b + F^5 = 0
\end{gathered}
$$

This proves that a quartic factors since it depends on factoring a sextic which factors because its Sylvester determinant has odd degree 15.

6.3 Octic Determinants

6.3.1 Elimination of b

$$
\begin{array}{rlr}
p(b) = T8 = & (4a+A)b^3 + (10a^3+6Aa^2+3Ba+C)b^2 + (6a^5+5Aa^4+4Ba^3+3Ca^2+2Da+E)b + & \\
& a^7+Aa^6+Ba^5+Ca^4 + Da^3+Ea^2+Fa+G = 0 & (m = 3) \\
q(b) = H+bT7 = & b^4 + (6a^2+3Aa+B)b^3 + (5a^4+4Aa^3+3Ba^2+2Ca+D)b^2 + & \\
& (a^6+Aa^5+Ba^4+Ca^3+Da^2+Ea+F)b + H = 0 & (n = 4)
\end{array}
$$

$$
\begin{array}{llll}
b^6 & b^5 & b^4 & b^3
\end{array}
$$
$$
\left|\begin{array}{llll}
1 & 6a^2+3Aa+B & 5a^4+4Aa^3+3Ba^2+2Ca + D & a^6+Aa^5+Ba^4+Ca^3+D^2+Ea+F \\
0 & 1 & 6a^2+3Aa+B & 5a^4+4Aa^3+3Ba^2+2Ca + D \\
0 & 0 & 1 & 6a^2+3Aa+B \\
4a+A & 10a^3+6Aa^2+3Ba+C & 6a^5+5Aa^4+Ba^3+Ca^2+Da+E & a^7+Aa^6+Ba^5+Ca^4+Da^3+Ea^2+Fa+G \\
0 & 4a+A & 10a^3+6Aa^2+3Ba+C & 6a^5+5Aa^4+Ba^3+Ca^2+Da+E \\
0 & 0 & 4a+A & 10a^3+6Aa^2+3Ba+C \\
0 & 0 & 0 & 4a+A
\end{array}\right.
$$

b^2	b^1	b^0	
H	0	0	
$a^6+Aa^5+Ba^4+...+Ea+F$	H	0	
$5a^4+4Aa^3+3Ba^2+2C+D$	$a^6+Aa^5+Ba^4+Ca^3+D^2+Ea+F$	H	
0	0	0	$=$
$a^7+Aa^6+Ba^5+...+Ea^2+Fa+G$	0	0	
$6a^5+5Aa^4+Ba^3+Ca^2+Da+E$	$a^7+Aa^6+Ba^5+...+Ea^2+Fa+G$	0	
$10a^3+6Aa^2+3Ba+C$	$6a^5+5Aa^4+Ba^3+Ca^2+Da+E$	$a^7+Aa^6+Ba^5+...+Ea^2+Fa+G$	

$$a^{28} + 7Aa^{27} + (21A^2 + 6B)a^{26} + (35A^2 + 36AB + 4C)a^{25} + (35A^4 + 90A^2B + 15B^2 + 25AC)a^{24} + \\ (21A^5 + 120A^3B + 75AB^2 + (65A^2 + 20B)C + 4AD - 8E)a^{23} + \cdots + (...)a +$$

$$A^4H^3 + ((4A^2C - 3A^3B)G - A^3CF + 2A^2E^2 + (-2A^3D - 4AC^2 + 3A^2BC)E + A^2C^2D + C^4 - ABC^3)H^2 + \\ ((4AE - 3A^2D + 2C^2 - 5ABC + 3A^2B^2)G^2 + ((-5A^2E + 3A^3D + AC^2 - A^2BC)F + (AB - 4C)E^2 + \\ ((4AC + A^2B)D + 3BC^2 - 3AB^2)E - 2A^2CD^2 + (2ABC^2 - 2C^3)D)G + A^3EF^2 + \\ ((3AC - 2A^2B)E^2 + (-A^2CD - C^3 + ABC^2)E)F + E^4 + (-2AD - BC + AB^2)E^3 + \\ (A^2D^2 + (C^2 - ABC)D)E^2)H + G^4 + (-3AF - BE + (3AB - 2C)D + B^2C - AB^3)G^3 + (3A^2F^2 + \\ ((3C - AB)E + (AC - 3A^2B)D - 2BC^2 + 2AB^2C)F + DE^2 + ((AB^2 - BC)D - 2AD^2)E + A^2D^3 + \\ (C^2 - ABC)D^2)G^2 + (-A^3F^3 + ((2A^2B - 3AC)E + A^2CD + C^3 - ABC^2)F^2 + \\ (-E^3 + (2AD + BC - AB^2)E^2 + ((ABC - C^2)D - A^2D^2)E)F)G = 0$$

6.3.2 Elimination of a

$$p(a) = T8 = \quad a^7+Aa^6+(6b+B)a^5+(5Ab+C)a^4+(10b^2+4Bb+D)a^3+ \\ (6Ab^2+3Cb+E)a^2 + (4b^3+3Bb^2+2Db+F)a+Ab^3+Cb^2+Eb+G=0 \quad (m=7)$$

$$q(a) = H+bT7 = \quad ba^6+Aba^5+(5b^2+Bb)a^4+(4Ab^2+Cb)a^3+ \\ (6b^3+3Bb^2+Db)a^2+(3Ab^3+2Cb^2+Eb)a+b^4+Bb^3+Db^2+Fb+H=0 \quad (n=6)$$

a^{12}	a^{11}	a^{10}	a^9	a^8	a^7	a^6	a^5
1	A	$6b+B$	$5Ab+C$	$10b^2+4Bb+D$	$6Ab^2+3Cb+E$	$4b^3+3Bb^2+2Db+F$	Ab^3+Cb^2+Eb+G
0	1	A	$6b+B$	$5Ab+C$	$10b^2+4Bb+D$	$6Ab^2+3Cb+E$	$4b^3+3Bb^2+2Db+F$
0	0	1	A	$6b+B$	$5Ab+C$	$10b^2+4Bb+D$	$6Ab^2+3Cb+E$
0	0	0	1	A	$6b + B$	$5Ab + C$	$10b^2+4Bb+D$
0	0	0	0	1	A	$6b+B$	$5Ab+C$
0	0	0	0	0	1	A	$6b+B$
b	Ab	$5b^2+Bb$	$4Ab^2+Cb$	$6b^3+3Bb^2+Db$	$3Ab^3+2Cb^2+Eb$	b^4+Bb^3+Fb+H	0
0	b	Ab	$5b^2+Bb$	$4Ab^2+Cb$	$6b^3+3Bb^2+Db$	$3Ab^3+2Cb^2+Eb$	b^4+Bb^3+Fb+H
0	0	b	Ab	$5b^2+Bb$	$4Ab^2+Cb$	$6b^3+3Bb^2+Db$	$3Ab^3+2Cb^2+Eb$
0	0	0	b	Ab	$5b^2+Bb$	$4Ab^2+Cb$	$6b^3+3Bb^2+Db$
0	0	0	0	b	Ab	$5b^2+Bb$	$4Ab^2+Cb$
0	0	0	0	0	b	Ab	$5b^2+Bb$
0	0	0	0	0	0	b	Ab

a^4	a^3	a^2	a^1	a^0	
0	0	0	0	0	
Ab^3+Cb^2+Eb+G	0	0	0	0	
$4b^3+3Bb^2+2Db+F$	Ab^3+Cb^2+Eb+G	0	0	0	
$6Ab^2+3Cb+E$	$4b^3+3Bb^2+2Db+F$	Ab^3+Cb^2+Eb+G	0	0	
$10b^2+4Bb+D$	$6Ab^2+3Cb+E$	$4b^3+3Bb^2+2Db+F$	Ab^3+Cb^2+Eb+G	0	
$5Ab+C$	$10b^2+4Bb+D$	$6Ab^2+3Cb+E$	$4b^3+3Bb^2+2Db+F$	Ab^3+Cb^2+Eb+G	
0	0	0	0	0	$=$
0	0	0	0	0	
b^4+Bb^3+Fb+H	0	0	0	0	
$3Ab^3+2Cb^2+Eb$	b^4+Bb^3+Fb+H	0	0	0	
$6b^3+3Bb^2+Db$	$3Ab^3+2Cb^2+Eb$	b^4+Bb^3+Fb+H	0	0	
$4Ab^2+Cb$	$6b^3+3Bb^2+Db$	$3Ab^3+2Cb^2+Eb$	b^4+Bb^3+Fb+H	0	
$5b^2+Bb$	$4Ab^2+Cb$	$6b^3+3Bb^2+Db$	$3Ab^3+2Cb^2+Eb$	b^4+Bb^3+Fb+H	

$-5b^4+H$	$-5Ab^4-Gb$	$-16b^5-5Bb^4+Fb^2$	$-11Ab^5-5Cb^4-Eb^3+Gb^2$	$-8b^6-6Bb^5-4Db^4-2Fb^3$	$-2Ab^6-2Cb^5-2Eb^4-2Gb^3$	0
0	$-5b^4+H$	$-5Ab^4-Gb$	$-16b^5-5Bb^4+Fb^2$	$-11Ab^5-5Cb^4-Eb^3+Gb^2$	$-8b^6-6Bb^5-4Db^4-2Fb^3$	$-2Ab^6-2Cb^5-2Eb^4-2Gb^3$
$2b^3$	$2Ab^3$	$7b^4+2Bb^3+H$	$5Ab^4+2Cb^3-Gb$	$4b^5+3Bb^4+2Db^3+Fb^2$	$Ab^5+Cb^4+Eb^3+Gb^2$	0
0	$2b^3$	$2Ab^3$	$7b^4+2Bb^3+H$	$5Ab^4+2Cb^3-Gb$	$4b^5+3Bb^4+2Db^3+Fb^2$	$Ab^5+Cb^4+Eb^3+Gb^2$
$-b^2$	$-Ab^2$	$-4b^3-Bb^2$	$-3Ab^3-Cb^2$	$-3b^4-2Bb^3-Db^2+H$	$-Ab^4-Cb^3-Eb^2-Gb$	0
0	$-b^2$	$-Ab^2$	$-4b^3-Bb^2$	$-3Ab^3-Cb^2$	$-3b^4-2Bb^3-Db^2+H$	$-Ab^4-Cb^3-Eb^2-Gb$
b	Ab	$5b^2+Bb$	$4Ab^2+Cb$	$6b^3+3Bb^2+Db$	$3Ab^3+2Cb^2+Eb$	$b^4+Bb^3+Db^2+Fb+H$

$$b^{28} + Bb^{27} + (AC - D)b^{26} + (F - AE + (A^2 - 2B)D + C^2)b^{25} +$$
$$(-H + AG + (2B - A^2)F + (C - 3AB + A^3)E - D^2 + ACD)b^{24} +$$
$$((A^2-2B)H+(-C+3AB-A^3)G+(-2D+3AC+2B^2-4A^2B+A^4)F+2E^2+((A^2-2B)C-2AD)E+BD^2)b^{23}+$$

$$\cdots + (\cdots)b + (-H^6 + (AG + 2BF + CE - D^2)H^5 + ((DE - 3CF)G - BG^2)H^4 + CGH^3)b^4 +$$
$$(BH^6 + (-CG - 2DF + E^2)H + DG^2H^4)b^3 + (EGH^5 - DH^6)b^2 + FH^6b + H^7 = 0$$

6.4 Decic Determinants

6.4.1 Elimination of b

$$\begin{aligned}
p(b)=T10= \quad & I + aT9 + bT8 = I + a(H + aT8 + bT7) + bT8 = I + aH + (a^2 + b)T8 + abT7 = \\
& (5a^4 + A)b^4 + (20a^3+10Aa^2+4Ba+C)b^3+ \\
& (21a^5+15Aa^4+10Ba^3+6Ca^2+3Da+E)b^2 + \\
& (8a^7 + 7a^6 + 6Ba^5 + 5Ca^4 + 4Da^3 + 3Ea^2 + 2Ea^2 + 2Fa + G)b+ \\
& a^9+Aa^8+Ba^7+Ca^6 + Da^5+Ea^4+Fa^3+Ga^2+ Ha + I =0 \qquad (m = 4) \\
q(b)=J+bT9= \quad & b^5+(10a^2+4Aa+B)b^4+(15a^4+10Aa^3+6Ba^2+3Ca+D)b^3+ \\
& (7a^6+6Aa^5+5Ba^4+4Ca^3+3Da^2+2Ea+F)b^2+ \\
& a^8+Aa^7+Ba^6+Ca^5 + Da^4+Ea^3+Fa^2+Ga+ H)b + J = 0 \qquad (n = 5)
\end{aligned}$$

D(a) is a degree 9 Sylvester determinant whose 9 columns may be imagined as labelled $b^8 \cdots b^0$. It's first 5 rows are the successive coefficients of the powers of b from $p(b)$ padded with zeros successively shifted. The remaining 4 rows are the coefficients of the powers of b in $q(b)$ padded with zeros successively shifted.

6.4.2 Elimination of a

$$\begin{aligned}
p(a) =T10= \quad & a^9+Aa^8+(8b+B)a^7+(7Ab+C)a^6+(21b^2+6Bb+D)a^5+ \\
& (15Ab^2+5Cb+E)a^4 + (20b^3+10Bb^2+4Db+F)a^3 + (10Ab^3+6Cb^2+3Eb+G)a^2+ \\
& (5b^4+4Bb^3+3Db^2+2Fb+H)a + Ab^4 + Cb^3 + Eb^2 + Gb + I=0 \qquad (m = 9) \\
q(a) =J+bT9= \quad & ba^8+Aba^7+(7b^2+Bb)a^6+(6Ab^2+Cb)a^5+ \\
& (15b^3+5Bb^2+Db)a^4+(10Ab^3+4Cb^2+Eb)a^3+(10b^4+6Bb^3+3Db^2+Fb)a^2+ \\
& (4Ab^4 + 3Cb^3 + 2Eb^2 + Gb)a+ b^5 + Bb^4 + Db^3 + Fb^2 + Hb + J = 0 \qquad (n = 8)
\end{aligned}$$

D(b) is a degree 17 Sylvester determinant whose 17 columns may be imagined as labelled $a^{16} \cdots a^0$. It's first 8 rows are the successive coefficients of the powers of a from $p(a)$ padded with zeros successively shifted. The remaining 9 rows are the coefficients of the powers of a in $q(a)$ padded with zeros successively shifted.

7 Clifford's Determinant Process

The key to understanding Clifford's process is what **one more degree odd** means in his abstract which is here edited for clarity with Clifford's s replaced by n. "Thus the determination of a quadratic factor of an expression of degree $2n$ is reduced to the solution of an equation of order $n(2n-1)$ (via Professor Sylvester's Dialytic method)[sic]. But this number is one more degree odd than the original number; that, is to say, if the number $2n$ is 2^k multiplied by an odd number, say o, so $2n = 2^k o$, then $n(2n-1) = \frac{1}{2}2^k o(2^k o - 1) = 2^{k-1}o(2^k o - 1)) = 2^{k-1}$ multiplied by an odd number ($= o(2^k o - 1)$)"; so a step of the process moves from an equation of degree $2^k o$ to one of degree $2^{k-1}o(2^k o - 1)$. This shows two things: both the degree and odd number increase. The phrase **one more degree odd** means the number of odd factors has increased by one. Indeed, the process stops in k steps at the final determinant of odd degree of k odd factors.

The following two tables of D(b)'s degrees help to see the process steps.

k	2^k1	2^k3	2^k5	2^k7	2^k9	2^k11	2^k13
1	$2 \to 1$	$6 \to 15$	$10 \to 45$	$14 \to 91$	$18 \to 153$	$22 \to 231$	$26 \to 235$
2	$4 \to 6$	$12 \to 66$	$20 \to 190$	$28 \to 378$	$36 \to 630$	$44 \to 7946$	52
3	$8 \to 28$	$24 \to 276$	$40 \to 780$	56	72	88	104
4	$16 \to 120$	$48 \to 1128$	80	112	144	178	208
5	$32 \to 496$	64	160	224	288	356	416

$4 \to 6 \to 15$
$6 \to 15$
$8 \to 28 \to 378 \to 71253$
$10 \to 45$
$12 \to 66 \to 2145$
$14 \to 91$
$16 \to 120 \to 7140 \to 25486230 \to 12743115 \times 25486229$
$18 \to 153$
$20 \to 190 \to 17955$
$22 \to 231$
$24 \to 276 \to 37950 \to 720082275$

8 Factoring Via Sylvester Determinants

Clifford's FTA existence proof implies initial Sylvester determinants $\mathrm{D}(a)=0$ and $\mathrm{D}(b)=0$ have solutions giving factors but this does not mean they will be easy to find. For example, $\mathrm{D}(b)=0$ for x^8+1 is a cubic in a^8 which does not have an obvious algebraic solution at first inspection.

The determinants $\mathrm{D}(b)$'s for the quartic, sextic, octic, and decic expressions have positive constant terms D^3, F^5, H^7, and J^9, respectively. This is not true for determinants $\mathrm{D}(a)$. For example, quartic $\mathrm{D}(a)$'s constant equals zero if $A=C=D=1$ and $B=2$. It's better to eliminate b than a because deg $\mathrm{D}(a)$ may be strictly less than deg $\mathrm{D}(b)$. Moreover, $\mathrm{D}(a)$'s constant maybe negative so the process stops by the intermediate value theorem. If the constant term $A(BC-AD)-C^2$ of the quartic Sylvester determinant is positive, the process continues. If the constant is zero, it might be that $\mathrm{D}(0)=0$; i.e. a quadratic factor x^2-b occurs or not. For the example here, $b=-1$ because $p(-b)=q(-b)=0$ and the quartic factors as $x^4+x^3+2x^2+x+1=(x^2+1)(x^2+x+1)$. $\mathrm{D}(a)=0$ may have solutions for which $\mathrm{D}(b)\neq 0$ and vice versa $\mathrm{D}(b)=0$ may have solutions for which $\mathrm{D}(a)\neq 0$. Only a and b pairs for which $\mathrm{D}(a)$=$\mathrm{D}(b)=0$ simultaneously are valid.

8.1 $x^4+1 \Rightarrow A=B=C=0, D=1$

8.1.1 Elimination of b

$$\begin{vmatrix} 1 & a^2 & 1 \\ 2a & a^3 & 0 \\ 0 & 2a & a^3 \end{vmatrix} = -a^2(a^4-4) = 0 \Rightarrow a{=}0, \pm\sqrt{2}$$

$$\begin{array}{llll} \text{for } a{=}\sqrt{2} & p(b){=}2ab{+}a^3{=}0 & {=}2\sqrt{2}b{+}2\sqrt{2} \Rightarrow b{=}{-}1 & \Rightarrow \quad x^2{-}ax{-}b{=}x^2{-}\sqrt{2}x{+}1 \\ \text{for } a{=}{-}\sqrt{2} & q(b){=}b^2{+}a^2b{+}1{=}0 & {=}b^2{+}2b{+}1{=}(b{+}1)^2 \Rightarrow b{=}{-}1 & \Rightarrow \quad x^2{-}ax{-}b{=}x^2{+}\sqrt{2}x{+}1 \end{array}$$

$$x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$$

8.1.2 Elimination of a

$$\begin{vmatrix} 1 & 0 & 2b & 0 & 0 \\ 0 & 1 & 0 & 2b & 0 \\ b & 0 & b^2+1 & 0 & 0 \\ 0 & b & 0 & b^2+1 & 0 \\ 0 & 0 & b & 0 & b^2+1 \end{vmatrix} = b^6-b^4-b^2+1{=}\,(b^2-1)^2(b^2+1) = 0 \Rightarrow b{=}{\pm}1$$

$$\begin{array}{llll} \text{for } b{=}{-}1 & p(b){=}a(2b{+}a^2)=0 & {=}a({-}2{+}a^2) \Rightarrow a{=}0, \pm\sqrt{2} & \Rightarrow \quad x^2{-}ax{-}b{=}x^2{-}\sqrt{2}x{+}1 \\ \text{for } b{=}{-}1 & q(b){=}b^2{+}a^2b{+}1{=}0 & {=}1{-}a^2{+}1{=}(2{-}a^2) \Rightarrow a{=}{\pm}\sqrt{2} & \Rightarrow \quad x^2{-}ax{-}b{=}x^2{+}\sqrt{2}x{+}1 \end{array}$$

$$x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$$

8.2 $x^6+1 \Rightarrow A=B=C=D=E=0, F=1$

8.2.1 Elimination of b

$$\begin{vmatrix} 1 & 3a^2 & a^4 & 1 & 0 \\ 0 & 1 & 3a^2 & a^4 & 1 \\ 3a & 4a^3 & a^5 & 0 & 0 \\ 0 & 3a & 4a^3 & a^5 & 0 \\ 0 & 0 & 3a & 4a^3 & a^5a \end{vmatrix} = a^{15}-26a^{12}-27a^3{=}\,a^3(a^6+1)(a^6-27) = 0 \Rightarrow a{=}0, \pm\sqrt{3}$$

$$x^6+1=(x^2+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)$$

8.2.2 Elimination of a

$$\begin{vmatrix} 1 & 0 & 4b & 0 & 3b^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4b & 0 & 3b^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4b & 0 & 3b^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4b & 0 & 3b^2 & 0 \\ b & 0 & 3b^2 & 0 & b^3+1 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 3b^2 & 0 & b^3+1 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 3b^2 & 0 & b^3+1 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 3b^2 & 0 & b^3+1 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 3b^2 & 0 & b^3+1 \end{vmatrix} = b^{15}+b^{12}-2b^9-2b^6+b^3+1=0 \Rightarrow b=\pm 1$$

for $b{=}{-}1$ $p(a)=a^5-4a^3+3a=a((a^2-1)(a^2-3)=0 \Rightarrow a=0, \pm 1, \pm\sqrt{3}$
for $b=-1$ $q(a)=-a^4+3a^2+-1+1=-a^2(a^2-3)=0 \Rightarrow a=0, \pm\sqrt{3}$

$$x^6+1=(x^2+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)$$

8.3 $x^8 + 1 \Rightarrow A = B = C = D = E = F = G = 0, H = 1$

8.3.1 Elimination of b

$$\begin{vmatrix} 1 & a^2 & 5a^4 & a^6 & 1 & 0 & 0 \\ 0 & 1 & a^2 & 5a^4 & a^6 & 1 & 0 \\ 0 & 0 & 1 & a^2 & 5a^4 & a^6 & 1 \\ 4a & 10a^3 & 6a^5 & a^7 & 0 & 0 & 0 \\ 0 & 4a & 10a^3 & 6a^5 & a^7 & 0 & 0 \\ 0 & 0 & 4a & 10a^3 & 6a^5 & a^7 & 0 \\ 0 & 0 & 0 & 4a & 10a^3 & 6a^5 & a^7 \end{vmatrix} = a^4((a^8)^3 - 120(a^8)^2 - 2160a^8 + 256) = 0 \Rightarrow a{=}0, ????$$

The cubic in a^8 doesn't present an easily found analytic solution, but fortunately we can find it by recognizing that

$$x^8 + 1 = (x^2)^4 + 1 = (x^4 + \sqrt{2}x^2 + 1)(x^4 - \sqrt{2}x^2 + 1)]$$

from 8.1 and now find the factors of the quartics

$$x^4 + \sqrt{2}x^2 + 1 \text{ and } x^4 - \sqrt{2}x^2 + 1.$$

To this end we see $x^4 \pm \sqrt{2}x^2 + 1 \Rightarrow A = C = 0, B = \pm\sqrt{2}, D = 1$

8.3.1.1 Elimination of b

$$\begin{vmatrix} 1 & a^2 \pm \sqrt{2} & 1 \\ 2a & a^3 \pm \sqrt{2}a & 0 \\ 0 & 2a & a^3 \pm \sqrt{2}a \end{vmatrix} = a^2(a^2 \pm \sqrt{2})^2 - 2a^2((a^2 \pm \sqrt{2})^2 - 2) = -a^2(a^2 \pm \sqrt{2})^2 + 4a^2 = 0$$

$$\Rightarrow a = \pm\sqrt{2 \pm \sqrt{2}}$$

8.3.1.2 Elimination of a

$$\begin{vmatrix} 1 & 0 & 2b \pm \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 2b \pm \sqrt{2} & 0 \\ b & 0 & b^2 \pm \sqrt{2}b + 1 & 0 & 0 \\ 0 & b & 0 & b^2 \pm \sqrt{2}b + 1 & 0 \\ 0 & 0 & b & 0 & b^2 \pm \sqrt{2}b + 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2b \pm \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 2b \pm \sqrt{2} & 0 \\ 0 & 0 & -b^2 + 1 & 0 & 0 \\ 0 & 0 & 0 & -b^2 + 1 & 0 \\ 0 & 0 & b & 0 & b^2 \pm \sqrt{2}b + 1 \end{vmatrix} =$$

$$(-b^2 + 1)^2(b^2 \pm \sqrt{2} + 1) = (-b^2 + 1)^2((b \pm \sqrt{2}/2)^2 + 1/2) = 0$$

$$\text{for } b = -1, p(a) = a^3 + a(2b \pm \sqrt{2}a) = 0 \Rightarrow a^2 = -2 \pm \sqrt{2} \Rightarrow a = \pm\sqrt{(2 \pm \sqrt{2})}$$

$$\text{for } b = -1, q(a) = ba^2 + b^2 \pm \sqrt{2}b + 1 = 0 \Rightarrow -a^2 + 1 \pm \sqrt{2}b + 1 = 0 \Rightarrow a = \pm\sqrt{(2 \pm \sqrt{2})}$$

$$x^8 + 1 = (x^2 - \sqrt{2 + \sqrt{2}}x + 1)(x^2 + \sqrt{2 + \sqrt{2}}x + 1)(x^2 - \sqrt{2 - \sqrt{2}}x + 1)(x^2 + \sqrt{2 - \sqrt{2}}x + 1)$$

so $a = \pm\sqrt{2 - \sqrt{2}}, \pm\sqrt{2 + \sqrt{2}}$ and the cubic has the solution $a^8 = (\sqrt{2 \pm \sqrt{2}})^8 = 4(17 \pm 12\sqrt{2})$ resulting in

$$\begin{aligned} &y^3 - 120y^2 - 2160y + 256 = \\ &(y - 4(17 + 12\sqrt{2}))(y^2 - 4(13 - 12\sqrt{2})y - (1088 - 768\sqrt{2})) = \\ &(y - 4(17 - 12\sqrt{2}))(y^2 - 4(13 + 12\sqrt{2})y - (1088 + 768\sqrt{2})) = \\ &(y + 16)(y - 4(17 + 12\sqrt{2}))(y - 4(17 - 12\sqrt{2})) = 0. \end{aligned}$$

8.3.2 Elimination of a

$$\begin{vmatrix} 1, 0, 6b, 0, 10b^2, 0, b^3, 0, 0, 0, 0, 0, 0 \\ 0, 1, 0, 6b, 0, 10b^2, 0, b^3, 0, 0, 0, 0, 0 \\ \text{etc.} \\ 0, 0, 0, 0, 0, 1, 0, 6b, 0, 10b^2, 0, b^3, 0 \\ b, 0, 5b^2, 0, 6b^3, 0, b^4 + 1, 0, 0, 0, 0, 0, 0 \\ 0, b, 0, 5b^2, 0, 6b^3, 0, b^4 + 1, 0, 0, 0, 0, 0 \\ \text{etc.} \\ 0, 0, 0, 0, 0, 0, b, 0, 5b^2, 0, 6b^3, 0, b^4 + 1 \end{vmatrix} = b^{28} - b^{24} - 3b^{20} + 3b^{16} + 3b^{12} - 3b^8 - b^4 + 1 = 0 \Rightarrow b = \pm 1$$

for $b = -1$ $q(a) = a^6 - 5a^4 + 6a^2 - 2 = (a^2 - 1)(a^4 - 4a^2 + 2) = 0 \Rightarrow a^4 - 4a^2 + 2 = (a^2 - 2)^2 - 2 = 0 \Rightarrow a = \pm\sqrt{2 \pm \sqrt{2}}$ for $b = -1$ $p(a) = a^7 - 6a^5 + 10a^3 - 4a = a(a^6 - 6a^4 + 10a^2 - 4) = 0$ because $a(a^6 - 6a^4 + 10a^2 - 4) - a^3(a^4 - 4a^2 + 2) = -2a(a^4 - 4a^2 + 2) = 0$ if $a \neq 0$ and $a = \pm\sqrt{2 \pm \sqrt{2}}$. Nota bene how easy is the solution for a in this elimination.

8.4 $x^{10} + 1 \Rightarrow A = B = C = D = E = F = G = H = I = 0, J = 1$

8.4.1 Elimination of b

$$\begin{aligned} p(b) &= b^5 + 10a^2b^4 + 15a^5b^3 + 7a^6b^2 + a^8 + 1 = 0 \\ q(b) &= 5ab^4 + 20a^3b^3 + 21a^5b^2 + 8a^7b + a^9 = 0 \end{aligned}$$

$$\begin{vmatrix} 1 & 10a^2 & 15a^4 & 7a^6 & a^8 & 1 & 0 & 0 & 0 \\ 0 & 1 & 10a^2 & 15a^4 & 7a^6 & a^8 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10a^2 & 15a^4 & 7a^6 & a^8 & 1 & 0 \\ 0 & 0 & 0 & 1 & 10a^2 & 15a^4 & 7a^6 & a^8 & 1 \\ 5a & 20a^3 & 21a^5 & 8a^7 & a^9 & 0 & 0 & 0 & 0 \\ 0 & 5a & 20a^3 & 21a^5 & 8a^7 & a^9 & 0 & 0 & 0 \\ 0 & 0 & 5a & 20a^3 & 21a^5 & 8a^7 & a^9 & 0 & 0 \\ 0 & 0 & 0 & 5a & 20a^3 & 21a^5 & 8a^7 & a^9 & 0 \\ 0 & 0 & 0 & 0 & 5a & 20a^3 & 21a^5 & 8a^7 & a^9 \end{vmatrix} =$$

$$a^5(a^4-5a^2+5)(a^4+3a^2+1)(a^8+10a^4+25a^2+25)(a^8-4a^6+6a^4+a^2+1)(a^8+a^6+6a^4-4a^2+1)(a^8+5a^6+10a^4+25)=0$$
$$\Rightarrow (a^4-5a^2+5)=((a^2)^2-5a^2+5)=((a^2-5/2)^2-25/4+5)=((a^2-5/2)^2-5/4)=$$
$$(a^2-5/2-\sqrt{5}/2)(a^2-5/2+\sqrt{5}/2)=(a\pm\frac{\sqrt{5-\sqrt{5}}}{\sqrt{2}})(a\pm\frac{\sqrt{5+\sqrt{5}}}{\sqrt{2}})=0$$
$$\text{for } a=0,\ p(b)=b^5+1=0\Rightarrow b=-1, b\neq 1$$

8.4.2 Elimination of a

$$p(a)=a^9+8a^7b+21a^5b^2+20a^3b^3+5ab^4=0$$
$$q(a)=a^8b+7a^6b^2+15a^3+10a^2b^4+b^5+1=0$$

$$\begin{vmatrix} 1,0,8b,0,21b^2,0,20b^3,0,5b^4 0,0,0,0,0,0,0,0 \\ 0,1,0,8b,0,21b^2,0,20b^3,0,5b^4 0,0,0,0,0,0,0 \\ 0,0,1,0,8b,0,21b^2,0,20b^3,0,5b^4 0,0,0,0,0,0 \\ \text{etc.} \\ 0,0,0,0,0,0,0,1,0,8b,0,21b^2,0,20b^3,0,5b^4,0 \\ b,0,7b^2,0,15b^3,0,10b^4,0,b^5+1,0,0,0,0,0,0,0,0 \\ 0,b,0,7b^2,0,15b^3,0,10b^4,0,b^5+1,0,0,0,0,0,0,0 \\ 0,0,b,0,7b^2,0,15b^3,0,10b^4,0,b^5+1,0,0,0,0,0,0 \\ \text{etc.} \\ 0,0,0,0,0,0,0,0,b,0,7b^2,0,15b^3,0,10b^4,0,b^5+1 \end{vmatrix} = \begin{vmatrix} 14b^5+1,0,70b^6,0,90b^7,0,25b^8,0 \\ 0,14b^5+1,0,70b^6,0,90b^7,0,25b^8 \\ -5b^4,0,1-26b^5,0,-35b^4,0,-10b^7,0 \\ 0,-5b^4,0,1-26b^5,0,-35b^4,0,-10b^7 \\ 2b^3,0,11b^4,0,16b^5,0,5b^6,0 \\ 0,2b^3,0,11b^4,0,16b^5,0,5b^6 \\ -b^2,0,-6b^3,0,-10b^4,0,1-4b^5,0 \\ 0,-b^2,0,-6b^3,0,-10b^4,0,1-4b^5 \end{vmatrix} (b^5+1)=$$
$$(b^{40}-4b^{30}+6b^{20}-4b^{10}+1)(b^5+1)=$$
$$(b-1)^4(b+1)^4(b^4-b^3+b^2-b+1)^4(b^4+b^3+b^2+b+1)^4(b^5+1)=$$
$$(b^{40}-1)(b^5+1)=0$$
$$\Rightarrow b=-1, b\neq 1$$
$$\Rightarrow x^{10}+1=(x^2+1)(x^2-\frac{\sqrt{5-\sqrt{5}}}{\sqrt{2}}x+1)(x^2+\frac{\sqrt{5-\sqrt{5}}}{\sqrt{2}}x+1)(x^2-\frac{\sqrt{5+\sqrt{5}}}{\sqrt{2}}x+1)(x^2+\frac{\sqrt{5+\sqrt{5}}}{\sqrt{2}}x+1)$$

See x^2+1 is a factor of $x^{10}+1=x^{2^5}+1$. Standard synthetic division given by the following tableau, shows that the complementary factor is $x^{2^4}-x^{2^3}+x^{2^2}-x^{2^1}+1$.

	1	0	0	0	0	1
−1		−1	1	−1	1	−1
	1	−1	1	−1	1	0

If we'd known this before hand, we could have used a quartic Sylvester determinant on powers of x^2 where $A=C=-1$ and $B=D=1$ ultimately arriving at the full five factor expression above. No doubt this is easier because the Sylvester determinant is degree 6 rather than 45. This happened in factoring x^8+1 where we traded degree 6 for 28.

9 Conclusion

In [5], we find: "The cover-up (of the FTA)[sic] had continued through college, and algebra's superstar theorem was obscure as ever." and "Clearly a complete proof *is* beyond the reach of elementary mathematics." Algebra teachers and text books promote the FTA as easy to state and undoubted by examples but difficult to prove. We claim these comments are false so every college algebra student deserves at least to hear of Clifford's (of Clifford Algebra and Circle fame) FTA Proof.

10 References

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