Optimization of Fitzpatrick Functions [1]

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Abstract. We consider optimization problems for a class of convex functions on $H \times H$ introduced by Simon Fitzpatrick, where H is a real Hilbert space. We show that we can transform the minimization problem for a Fitzpatrick function on $H \times H$ into a simpler minimization problem on H by using the monotone operator created from the convex functional, introduced also by Fitzpatrick. We also present an iterative procedure for the solution. Finally, we give a new example of a Fitzpatrick function related to Lyapunov functions.

Keywords: Maximal monotone operator, lower semicontinuous convex maps, differential inclusions, optimization problems.

1 Introduction

Simon Fitzpatrick developed in [1] the representation of monotone operators on a normed space E in terms of the subdifferentials of convex functions on $E \times E^*$, which has led to definition of the Fitzpatrick functions (see the Definition 3.1 in Section 3).

Seminal ideas are often triggered by amazing insights. The Fitzpatrick functions represent an example of such idea. During the last two decades several mathematicians have published deep results considering the Fitzpatrick functions and their properties. The functions have played pivotal roles in advancing our understanding of maximal monotone operators and convex functions in three directions. Firstly, some results and connections have been found between Fitzpatrick functions and other results in the theory of monotone operators and convex analysis. Secondly, several deep results on maximal monotone operators have recently found simpler proofs using the Fitzpatrick functions. Thirdly, as demonstrated in this paper, the Fitzpatrick functions have interesting properties in optimization problems.

Penot [2] introduced new representations for maximal monotone operators and related them to previous representations given by Kraus, Fitzpatrick, Martinez-Legas, and Théra. He showed the usefulness of such representations for the study of compositions and sums of maximal monotone operators, highlighting the relevance of convex analysis for the study of monotonicity. Simons and Zalinescu [3] showed how the versions of the Fenchel duality theorem due to Rockafellar and Attouch-Brezis can be applied to the Fitzpatrick function determined by a maximal monotone multifunction to obtain a number of results on maximal monotonicity, including sufficient conditions for the sum of maximal monotone multifunctions on a reflexive Banach space to be a maximal monotone, unifying several results of the Attouch-Brezis type that have been obtained in recent years. Bartz et al. [4] studied a sequence of Fitzpatrick functions associated with a monotone operator. The first term of their sequence concides with the original Fitzparick function, and the other terms turn out to be useful for the identification and characterization of cyclic monotonicity properties. Borwein and Dutta [5] studied maximal monotone inclusions from the properties of gap functions. They proposed a natural gap function for an arbitrary maximal monotone inclusion, and showed how naturally this gap function arises from the Fitzpatrick function.

In the present paper, we consider optimization problems for the Fitzpatrick functions, utilizing specific properties such as maximal monotonicity, convexity and lower-semicontinuity. We generalize and unify preliminary results annouced mainly without proofs at two conferences. The paper also contains new material connecting Lyapunov functions with Fitzpatrick functions. Finally, we present new numerical results, which introduce simplified algorithms for this type of problems. The optimization of Fitzpatrick functions is motivated by considerations which can be best understood after presentation of the main results, and consequently they are deferred to concluding comments at the end of the paper. Concluding remarks present the main topics of the paper with possible future developments and applications.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let the norm of $H \times H$ be $\|(x, u)\| := \sqrt{\|x\|^2 + \|u\|^2}$, $(x, u) \in H \times H$. We assume that the convex functions are proper.

We recall that a mapping T of H into subsets of H is a *monotone operator* provided for each $u \in Tx$ and $v \in Ty$ the following inequality holds:

$$\langle u - v, x - y \rangle \geq 0.$$

If T is monotone and the *graph* of T is not properly contained in the graph of a monotone operator then T is said to be *maximal monotone.*

We remind also the next:

A vector $g \in H$ is a subgradient of a function $F : H \to R$ if for all $z \in DomF$ $F(z) \geq F(x) + \langle z - x, g \rangle$. The set of all subgradients of F at x is called the subdifferential of F at x and denoted by $\partial F(x)$. The subdifferential $\partial F(x)$ is always a closed convex set, even if F is not convex. This follows from the fact that it is an intersection of infinite set of half spaces:

$$\partial F(x) := \bigcap_{x \in DomF} \{g | F(z) \geq F(x) + \langle z - x, g \rangle\}.$$

It is well known that: if F is a proper convex lower semicntinuous function, then $\partial F : H \Longrightarrow H$ is a maximal monotone operator.

3 Minimization Problems for a Class Convex Functions on $H \times H$.

In this section we consider minimization problems for a class of convex functions on $H \times H$. We define and study a monotone operator on H using a convex function on $H \times H$, presented in [1].

Definition 3.1 *For each convex function f on $H \times H$ let*

$$T_f x := \{u \in H | (u, x) \in \partial f(x, u)\}$$

for each $x \in H$.

Fitzpatrick proved that T_f is a monotone operator (see [1] Proposition 2.2).

In this section we will consider next optimization problem

$$\inf_{(x,u) \in H \times H} f(x, u) = f^{inf} \tag{1}$$

supposing its solution exists, i.e. we suppose that there exists at least one point $(x^*, u^*) \in H \times H$ for which the solution of (1) exists.

Theorem 3.1 *Let f and T_f be as in Definition 3.1 and let function $f(x, u)$ is defined, proper, convex, and lower semicontinuous and T_f is maximal monotone.*

Then differential inclusions

$$\dot{x} \in -T_f(x(t)), \quad x(0) = x_0 \tag{2}$$

and

$$(\dot{x}, \dot{u}) \in -\partial f(x(t), u(t)), (x(0), u(0)) = (x_0, u_0) \tag{3}$$

have unique solutions and $x(\cdot)$ *and* $(x(\cdot), u(\cdot))$ *respectively and they coincide when* $(x, u) \in G(T_f)$

Proof. Because the function f is real valued, proper, lower semicontinuous, and convex on $H \times H$, then ∂f will be maximal monotone (see [8] Ch. 6, S. 7, Prop. 6). Since T_f is maximal monotone and ∂f is maximal monotone there exist unique solutions of differential inclusions (2) and (3), which follows from [8] (Ch. 6, S. 8, Theorem 1). We use the uniqueness of the solutions of (2) and (3) and the definitions of f and T_f.

It follows then that the solutions $x(.)$ of differential inclusion (2) and $(x(.), u(.))$ of differential inclusion (3) coincide for every $t \in [0, \infty)$ when $(x(t), u(t)) \in G(T_f)$ where T_f satisfies Definition 3.1.

□

In the rest of the section we consider a class of differential inclusions involving the subdifferential of the Fitzpatrick functions in which every solution of the differential inclusion can be extended up to the solution set.

Let

$X_f := \{(x, u) \in H \times H | f(x, u) = f^{inf}\}$, the solution set of the optimization problem (1) and

$X_{f_\varepsilon} := \{(x, u) \in H \times H | \min_{(y,v)} \|(x, u) - (y, v)\| < \varepsilon$ is an ε-neighborhood of X_f.

Lemma 3.1 *Consider the differential inclusion*

$$(\dot{x}, \dot{u}) \in (\Phi(x, u)), (x(0), u(0)) = (x_0, u_0),\ t \geq 0, \tag{4}$$

where $\Phi : H \times H \Longrightarrow R$ *is maximal monotone multi-valued map and therefore upper semicontinuous, and for each* $(x, u) \in H \times H$*,* $\Phi(x, u)$ *is bounded, closed, and convex set,* $\|\Phi(x, u)\| = \sup_{(w,z)\in\Phi(x,u)} \|(w, z)\| \leq 1 + \|(x, u)\|$*.*

Let $\theta(x, u) : H \times H \Longrightarrow R_+$ *be strictly positive and bounded function* $0 < \delta \leq \theta(x, u) \leq M < \infty$ *for which* $\theta(x(t), u(t))$ *is measurable for every absolutely continuous function* $(x(t), u(t))|R \Longrightarrow H \times H$*. Denote* $\tau(t) = \int_0^t \theta(x(s), u(s))ds$ *and* $t(\tau)$ *its inverse function. We consider also the following differential inclusion:*

$$(\dot{z}, \dot{w}) \in \theta(z, w)(\Phi(z, w)), (z(0), w(0)) = (x_0, u_0),\ t \geq 0 \tag{5}$$

If $(x(t), u(t))$ *is a solution of inclusion (4) then the function* $(z(t), w(t)) = (x(\tau(t)), u(\tau(t))$ *almost everywhere satisfies inclusion (5) i.e., the set of trajectories of differential inclusions* (4) *and* (5) *coincide as curves in the phase space* $H \times H$*.*

Proof. Note that, according to Deimling [9], there exists a solution of inclusion (4) if Φ maps bounded sets into bounded sets and satisfies some additional conditions of compactness. $\tau(t)$ is absolutely continuous function for which, under $0 < \delta \leq \theta(x, u)$, there exists an inverse function $t(\tau)$. Let $(x(t), u(t))$ be a solution of (4). The function

$$\tau(t) = \int_0^t \theta(x(s), u(s))ds$$

is absolutely continuous, and by $0 < \delta \le \theta(x,u) \le M < \infty$, it satisfies the Lipschitz condition with a constant M. Almost everywhere in t, for its derivative we have

$$\dot{\tau}(t) = \theta(x(t), u(t)) \ge \delta > 0.$$

Thus, the function $\tau(t)$ has an inverse function $t(\tau)$ for which

$$0 < \frac{1}{M} \le \dot{t}(\tau) = \frac{1}{\theta(x(t), u(t))} \le \frac{1}{\delta} < \infty.$$

Obviously, $t(\tau)$ is an absolutely continuous function. Now, almost everywhere in t we can write

$$\frac{d(x(\tau(t), u(\tau(t)))}{dt} \in \theta(x(\tau(t)), u(\tau(t)))\Phi(x(\tau(t), u(\tau(t))),$$

$$\frac{d(z(t(\tau)), w(t(\tau)))}{d\tau} \in \frac{1}{\theta(z(t(\tau)), w(t(\tau)))}\theta(z(t(\tau)), w(t(\tau)))\Phi(z(t(\tau)), w(t(\tau)))$$

$$= \Phi(z(t(\tau)), w(t(\tau)))$$

Thus, the positive multiplier $\theta(z,w)$ only transforms the time. □

We prove in the next theorem that if the optimization problem (1) has a solution, and therefore the correspondent differential inclusion (3) has a solution, then the solution of a modification of differential inclusion (3), satisfying Lemma 3.1, can be extended to the solution set X_f of (3), respectively of (1), in a finite time.

Recall that the metric projection of the point x in a Hilbert space H on a subset S is defined with $Pr_S x$, i.e.

$$Pr_S x := \{y \in S| \quad \|x - y\| = \min_{z \in S} \|x - z\|\}.$$

The metric projection on a closed and convex subset of a Hilbert space exists and is unique.

We denote here by $m(\mathbf{0}, \partial f(x,u))$ the metric projection of the origin onto $\partial f(x,u)$, that is,

$$m(\mathbf{0}, \partial f(x,u)) = \{(v,y) \in H \times H | \|(0,0) - (y,v)\| = \min_{(w,z)\in\partial f(x,u)} \|(0,0) - (w,z)\|\}.$$

We assume that $\mathbf{0}$ is not an element of the subdifferential.

We use also the notation

$$D(\mathbf{0}, \partial f(x,u)) = \sup_{(w,z)\in\partial f(x,u)} \|(0,0) - (w,z)\|.$$

In what follows when x and u depend on a parameter t(time), we will abbreviate the subdifferential in the metric projection and in the definition of D by $\partial f(t)$.

Theorem 3.2 *Let us consider again the differential inclusion*

$$(\dot{x}, \dot{u}) \in -\partial f(x(t), u(t)), (x(0), u(0)) = (x_0, u_0), \tag{3}$$

where $\partial f(x(.), u(.))$ is a maximal monotone operator. Suppose the map f is proper, convex and lower semicontinous, and maps (x,u) to a bounded, closed and convex set. For $\varepsilon > 0$ and $(x_0, u_0) \notin X_{f_\varepsilon}$ let $\|m(\mathbf{0}, \partial f(x,u))\| \geq \beta > 0,\ (x,u) \notin X_{f_\varepsilon}$ and let

$$X_f := \{(x,u) \in H \times H | f(x,u) = f^{inf}\} =$$

$$\{(z,w) \in H \times H | (0,0) \in \partial f(z,w)\}.$$

If the differential inclusion (3) *has a solution and the solutions set is bounded for $t \geq 0$, then every* *solution of*

$$(\dot{x},\dot{u}) \in -\frac{\partial f(x(t),u(t))}{\|m(\mathbf{0},\partial f(t))\|^2}, (x(0),u(0)) = (x_0,u_0) \tag{6}$$

can be extended up to the set X_f in a finite time $T \leq (f(x_0,u_0) - \underline{f})$ where $\underline{f}$ is a lower bound of $f(x,u)$

.

Proof. The function $\|m(\mathbf{0},\partial f(x,u))\|^2 \geq \beta^2 > 0$ is strictly positive and lower semicontinuous. Thus, for every absolutely continuous $(x(t),u(t))$ the function $\|m(\mathbf{0},\partial f(x(t),u(t)))\|$ is lower semicontinuous. From Lemma 3.1 the sets of trajectories of differential inclusions (3) and (6) coincide as curves in $H \times H$. Let $(x(t),u(t))$ be a solution of differential inclusion (6) and $\underline{f}$ is a lower bound of $f(x,u)$. The function $(f(x,u) - \underline{f})$ is real valued, strictly positive, proper, lower semicontinuous and convex. Suppose $f(x(t),u(t))$ is differentiable a.e. One writes

$$(f(x(0),u(0)) - \underline{f}) - (f(x(t),u(t)) - \underline{f})$$

$$= f(x(0),u(0)) - f(x(t),u(t))$$

$$= -\int_0^t (f(x(q),u(q)))'_q dq$$

$$\in -\int_0^t \big\langle \partial f(x(q),u(q)), (\dot{u}(q),\dot{x}(q)) \big\rangle dq$$

$$\subset \int_0^t \Big\langle \partial f(x(q),u(q)), \frac{\partial f(x(q),u(q))}{\|m(\mathbf{0},\partial f(q))\|^2} \Big\rangle dq$$

$$\subset t\Big[1, \frac{(D(\mathbf{0},\partial f(t)))^2}{\|m(\mathbf{0},\partial f(t))\|^2}\Big].$$

For every t we have $0 < f(x(t),u(t)) - \underline{f} \leq f(x(0),u(0)) - \underline{f} - t$. Thus in time $T \leq (f(x(0),u(0)) - \underline{f})$, where $\underline{f}$ is a lower bound of $f(x,u)$, the solution $(x(t),u(t))$ of (6), for $(x(0),u(0)) = (x_0,u_0)$, has to reach the set X_ε in time $T \leq (f(x(0),u(0)) - \underline{f}$, because the solutions set of inclusion (3) is bounded.

$\square$

Let us we consider the following theorem.

Theorem 3.3 *Consider the optimization problem*

$$\inf_{(x,u)\in H\times H} f(x,u) = f^{inf} \tag{1}$$

We suppose that f is proper, convex and lower semicontinuous, ∂f is maximal monotone, and denote by:

$X_f = \{(x,u) \in H\times H | f(x,u) = f^{inf}\}$, *the solution set of the optimization problem (1) and*

$X_{f_\varepsilon} = \{(x,u) \in H \times H | \min_{(y,v)} \|(x,u) - (y,v)\| < \varepsilon$ *is an ε-neighborhood of X_f.*

Let the solution set X_f be not empty and $(x_0,u_0) \notin X_{f_\varepsilon}$. Let $\|\partial f(x,u)\| \geq \gamma > 0$, $(x,u) \notin X_{f_\varepsilon}$.

Then:

1. The solution set X_f of (1) coincides with the set for which origin belongs to the subdifferential $\partial f(x,u)$,

2. For every initial position $(x_0,u_0) \in H \times H$ and every solutions $(x(t),u(t))$ of the differential inclusion

$$(\dot{x},\dot{u}) \in -\frac{\partial f(x(t),u(t))}{\|m(\mathbf{0},\partial f(t))\|^2}, \tag{6}$$

$$(x(0),u(0)) = (x_0,u_0),$$

where $m(\mathbf{0}, \partial f(t))$ *the metric projection of the origin onto the multivaued function* $\partial f(x,u)$*, that is,*

$$m(\mathbf{0}, \partial f(t)) = \{(v,y) \in E^* \times E | \|(0,0)-(y,v)\| = \min_{(w,z)\in\partial f(x(t),u(t))} \|(0,0)-(w,z)\|\},$$

there exists a finite moment $T \le f(x_0,u_0) - \underline{f}$*, where* $\underline{f}$ *is a lower bound of* $f(x,u)$*, for which* $(x(T),u(T))$ *solves the optimization problem (1).*

Proof. By the Lemma 3.1, the differential inclusions (6) and

$$(\dot{x}, \dot{u}) \in -\partial f(x(t),u(t)) \tag{3}$$

$$(x(0),u(0)) = (x_0,u_0)$$

have unique solutions which coincide as curves in $H \times H$ up to the moment T when $(0,0) \in \partial f(x(T),u(T))$.

We are going to show that the function $f(x,u)$ is the Lyapunov function for the differential inclusion (6). This method is based on the Lyapunov function theory. Under the Chain rule (see [10]) for the generalized gradientl of $f(x(t),u(t))$ almost everywhere in t we obtain

$$\partial_t f(x(t),u(t)) = \{\big\langle(\eta,\xi),(\dot{x}(t),\dot{u}(t))\big\rangle | (\eta,\xi) \in \partial f(y,v), (y,v) = (x(t),u(t)).$$

The differential inclusion (6) is the Filipov extension of the following differential equation with possibly discontinuous right-hand side, see [8] and [11] :

$$(\dot{x}, \dot{u}) = -\frac{m(\mathbf{0}, \partial f(t))}{\|m(\mathbf{0}, \partial f(t))\|^2}$$

$$(x(0),u(0)) = (x_0,u_0)$$

For every $(x(t),u(t))$ which satisfies the above differential equation and by the properties of the metric projection $m(\mathbf{0},\partial f(t))$ we have

$$\max_{(\eta,\xi)\in\partial f(x(t),u(t))}\big\langle(\eta,\xi),(\dot{x}(t),\dot{u}(t))\big\rangle=\max_{(\eta,\xi)\in\partial f(x(t),u(t))}\Big\langle(\eta,\xi),-\frac{m(\mathbf{0},\partial f(t))}{\|m(\mathbf{0},\partial f(t))\|^2}\Big\rangle=-1.$$

Thus, $f(x,u)$ is the Lyapunov function for the differential inclusion (6).

We obtain that $f(x(t),u(t))$ is a monotone decreasing function in t at the solutions of the differential inclusion (6).

The function $\|m(\mathbf{0},\partial f(x,u))\|^2\geq\gamma^2>0$ is strictly positive and lower semicontinuous. Thus, for every absolutely continuous $(x(t),u(t))$ the function $\|m(\mathbf{0},\partial f(x(t),u(t)))\|$ is lower semicontinuous. Under the Lemma 3.1 the sets of trajectories of differential inclusions (3) and (6) coincide as curves in $H\times H$. Let $(x(t),u(t))$ be a solution of differential inclusion (6) and $\underline{f}$ is a lower bound of $f(x,u)$. The function $f(x,u)-\underline{f}$ is real valued, strictly positive, proper, lower semicontinuous and convex. Assume $f(x(t),u(t))$ is differentiable a.e. One writes

$$(f(x(0),u(0))-\underline{f})-(f(x(t),u(t))-\underline{f})$$

$$=f(x(0),u(0))-f(x(t),u(t))$$

$$=-\int_0^t(f(x(q),u(q)))'_q dq$$

$$\in-\int_0^t\big\langle\partial f(x(q),u(q)),(\dot{x}(q),\dot{u}(q))\big\rangle dq$$

$$\subset\int_0^t\big\langle\partial f(x(q),u(q)),\frac{\partial f(x(q),u(q))}{\|m(\mathbf{0},\partial f(q))\|^2}\big\rangle dq$$

$$\subset t\Big[1, \frac{(D(\mathbf{0}, \partial f(t)))^2}{\|m(\mathbf{0}, \partial f(t))\|^2}\Big].$$

Here

$$D(\mathbf{0}, \partial f(t)) = \sup_{(w,z)\in\partial f(x(t),u(t))} \|(0,0) - (w,z)\|.$$

For every t, $q \in [0,t]$ we have $0 < f(x(t), u(t)) - \underline{f} \leq f(x(0), u(0)) - \underline{f} - t$. Thus in time $T \leq (f(x(0), u(0)) - \underline{f})$, where $\underline{f}$ is a lower bound of $f(x,u)$, the solution $(x(t), u(t))$ of (6), for $(x(0), u(0)) = (x_0, u_0)$, has to reach the set X_ε, because the solutions set of inclusion (3) is bounded.

Therefore according to [12] and Lemma 3.1 we can find a solution of the optimization problem (1) by solving differential inclusion (6). □

In the following theorem we denote with $m(\mathbf{0}, \Gamma_{T_f(t)})$ the metric projection of the origin onto the range of the multivalued map $T_f(x(t))$.

Theorem 3.4 *Let f and $T_f x$ satisfy Definition 3.1 and Theorem 3.1. Consider the differential inclusion*

$$\dot{x} \in -T_f(x(t)), \quad x(0) = x_0 \quad , \tag{2}$$

where the right-hand side is maximal monotone and maps x to a bounded, closed and convex set, $x_0 \notin X_{f_\varepsilon}$ and $\varepsilon > 0$ is arbitrary chosen. Let $\|m(\mathbf{0}, T_f(x))\| \geq \eta > 0$, $x \notin X_{f_\varepsilon}$ and

$$X_f := \{(x,u) \in H \times H | f(x,u) = f^{inf}\} = \{u \in T_f(x),\ (u,x) \in \partial f(x,u),\ |0 \in T_f(x)\}.$$

If the differential inclusion (2) admits a solution and the set of these solutions is bounded for $t \geq 0$, then every solution of

$$\dot{x} \in -\frac{T_f(x(t))}{\|m(\mathbf{0}, \Gamma_{T_f(t)})\|^2}, \quad x(0) = x_0,$$

can be extended up to the set X_f in a finite time $T \leq f(x_0, u_0) - \underline{f}$, where $u_0 \in T_f(x_0)$.

We omit the proof, because it is similar to the proof of Theorem 3.3.

4 Minimization Problem for a Fitzpatrick Function

Let us consider now next Fitzpatrick function.

Definition 4.1 *(see [1] 3.1. Definition) Let $T : H \to H$ be a monotone operator.*

$$L_T : H \times H \Longrightarrow] - \infty, +\infty] := (x, u) \mapsto \sup_{y \in H} \{\langle u, y\rangle + \langle Ty, x - y\rangle\}.$$

The function $L_T(x, u)$ is called the *Fitzpatrick function* representing a monotone operator T.

It follows from Definition 3.1 that if $D(T) \neq \emptyset$, then the function L_T is proper, lower semicontinuous and convex on $H \times H$ and ∂L_T is maximal monotone (see [6] and [7]).

According to Theorem 3.10 in [1] if T is a maximal monotone operator on H, then L_T is the minimal convex function f on $H \times H$ such that $f(x, u) \geq \langle u, x\rangle$ for all x and u, and $f(y, v) = \langle v, y\rangle$ for all $(y, v) \in G(T)$.

We will consider two cases.

Case 1

The following result will be used in what follows.

Theorem 4.1 *(see [1] 3.4. Theorem) If T is a monotone operator from Definition 4.1 on H and $(x, u) \in G(T)$ then $L_T(x, u) = \langle u, x\rangle$ and $(u, x) \in \partial L_T(x, u)$*

We can apply for this case directly Theorem 3.1 from the previous section because when $(x, u) \in G(T)$ then $(u, x) \in \partial L_T(x, u)$.

Case 2

The folloowing result will be used in what follows.

Lemma 4.1 *(see [1] 3.3. Lemma) If T is the monotone operator from Definition 4.1 on H and $(y, v) \in G(T)$ and for some $(x, u) \in H \times H$ we have*

$$L_T(x, u) = \langle v, x - y \rangle + \langle u, y \rangle$$

then $(y, v) \in \partial L_T(x, u)$.

Using Lemma 4.1 we prove the following theorem:

Theorem 4.2 *Let T and L_T be as in Definition 4.1, $D(T) \neq \emptyset$ and T is maximal monotone. And let there exist an open set $S \subset H \times H$ such that for every $(x, u) \in S$ the conditions of Lemma 4.1 will be satisfied and therefore if $(y, v) \in G(T)$, then $(y, v) \in \partial L_T(x, u)$. Suppose $(x, u) \in S$. Then the differential inclusion*

$$\dot{y} \in -T(y(t)), \quad y(0) = y_0 \tag{7}$$

has a unique solution $y(\cdot)$ defined on $[0, \infty)$. And the differential inclusion

$$(\dot{x}, \dot{u}) \in -\partial L_T(x(t), u(t)), (x(0), u(0)) = (x_0, u_0) = (y(0), v(0)) = (y_0, v_0) \tag{8}$$

has also a unique solution $(x(\cdot), u(\cdot))$ defined on $[0, \infty)$. The solutions $y(\cdot)$ of (7) and $(x(\cdot), u(\cdot))$ of (8) coincide when $(y, v) \in G(T)$ and $(x, u) \in S$.

Proof. Since the function L_T is real valued, proper, lower semicontinuous and convex on $H \times H$, the subdifferential ∂L_T is maximal monotone (see [8], Proposition 6, Section 6.7). Because T is maximal monotone and ∂L_T is maximal monotone the existence of the unique solutions of differential inclusions (7) and (8) follows from Theorem 1 Section 6.8 of [8]. Next we involve the uniqueness of the solutions of (7) and (8) and the definitions of T and L_T.

According to Lemma 4.1 above we know that when $(y, v) \in G(T)$ and $(x, u) \in S$ then $(y, v) \in \partial L_T(x, u)$. Thus the solutions of the differential inclusions (7) and (8) coincide for every $t \in [0, \infty)$ for which $(y(t), v(t)) \in G(T)$ and also because $(y, v) \in \partial L_T(x, u)$ when $(x, u) \in S$. □

According to Theorem 4.2, for solving optimization problem

$$\inf_{(x,u)\in H\times H} L_T(x, u) = L_T^{inf}. \tag{9}$$

we can solve instead of differential inclusion (8) the easier differential inclusion (7).

In the rest of the section we consider a class of differential inclusions involving the subdifferential of the Fitzpatrick functions in which every solution of the differential inclusion can be extended up to the solution set.

Let $X := \{(x, u) \in H \times H | L_T(x, u) = L_T^{inf}\}$ be the solution set of the optimization problem (9) and $X_\varepsilon := \{(x, u) \in H \times H | \min_{(y,v)\in X} \|(x, u) - (y, v)\| < \varepsilon > 0\}$ be an ε-neighborhood of X.

Recall that the metric projection of the point x in a Hilbert space H on a subset S is defined with

$Pr_S x$, i.e.

$$Pr_S x := \{y \in S| \quad \|x - y\| = \min_{z \in S} \|x - z\|\}.$$

The metric projection on a closed and convex subset of a Hilbert space exists and is unique.

We denote here by $m(\mathbf{0}, \partial L_T(x, u))$ the metric projection of the origin onto $\partial L_T(x, u)$, that is,

$$m(\mathbf{0}, \partial L_T(x, u)) = \{(v, y) \in H \times H| \|(0, 0) - (y, v)\| = \min_{(w,z) \in \partial L_T(x,u)} \|(0, 0) - (w, z)\|\}.$$

We assume that $\mathbf{0}$ is not an element of the subdifferential.

We use also the notation

$$D(\mathbf{0}, \partial L_T(x, u)) = \sup_{(w,z) \in \partial L_T(x,u)} \|(0, 0) - (w, z)\|.$$

In what follows when x and u depend on a parameter t(time), we will abbreviate the subdifferential in the metric projection and in the definition of D by $\partial L_T(t)$.

Theorem 4.3 *Let us consider again the differential inclusion*

$$(\dot{x}, \dot{u}) \in -\partial L_T(x(t), u(t)), (x(0), u(0)) = (x_0, u_0), \tag{8}$$

where $\partial L_T(x(.), u(.))$ is a maximal monotone operator. Suppose the map L_T is proper, convex and lower semicontinous, and maps (x, u) to a bounded, closed and convex set. For $\varepsilon > 0$ and $(x_0, u_0) \notin X_\varepsilon$ let $\|m(\mathbf{0}, \partial L_T(x, u))\| \geq \beta > 0,\ (x, u) \notin X_\varepsilon$ and let

$$X := \{(x, u) \in H \times H| L_T(x, u) = L_T^{inf}\} =$$

$$\{(z, w) \in H \times H| (0, 0) \in \partial L_T(z, w)\}.$$

If the differential inclusion (8) *has a solution and the solutions set is bounded for* $t \geq 0$*, then every solution of*

$$(\dot{x}, \dot{u}) \in -\frac{\partial L_T(x(t), u(t))}{\|m(\mathbf{0}, \partial L_T(t))\|^2},$$

$$(x(0), u(0)) = (x_0, u_0)$$

can be extended up to the set X *in a finite time* $T \leq (L_T(x_0, u_0) - \underline{L_T})$ *where* $\underline{L_T}$ *a lower bound of* $L_T(x, x^*)$.

We omit the proof, because it is similar to the proof of Theorem 3.3.

5 Iterative Method

In this section we present an iterative procedure for solving the optimization problems given in Sections 3 and 4.

Definition 4.1, presented in [1], introduces a monotone operator on H by using a convex function on $H \times H$. We denote by $m(\mathbf{0}, \partial f_k)$ the metric projection of the origin onto the multivalued map $\partial f(x_k, u_k)$ and with $m(\mathbf{0}, T_{f_k})$ the metric projection of the origin onto the range of the multivalued map $\partial(\|T_f(x_k)\|^2)$

We replace the following iterative procedure

$$(x_{k+1}, u_{k+1}) = (x_k, u_k) - \frac{f(x_k, u_k) m(\mathbf{0}, \partial f_k)}{\|m(\mathbf{0}, \partial f_k)\|^2},\ k = 0, 1, \ldots. \tag{10}$$

which is based on the differential inclusion,

$$(\dot{x}, \dot{u}) \in -\frac{\partial f(x(t), u(t))}{\|m(\mathbf{0}, \partial f(t))\|^2},\ (x(0), u(0)) = (x_0, u_0) \tag{6}$$

with the following simpler iterative procedure:

$$x_{k+1} = x_k - \frac{\|T_f(x_k)\|^2 m(\mathbf{0}, T_{f\,k})}{\|m(\mathbf{0}, T_{f\,k})\|^2},\ u_k = T_f(x_k),\ k = 0, 1, \ldots. \tag{11}$$

based on differential inclusion:

$$\dot{x} \in -\frac{T_f(x(t))}{\|m(\mathbf{0}, T_f(t))\|^2}, \quad x(0) = x_0.$$

We let set X_{f_ε} be the ε-neighborhood of X_f, i.e.

$$X_{f_\varepsilon} := \{(x, u) \in H \times H \mid \min_{(y,v)\in X_f} \| (x, u) - (y, v) \| < \varepsilon\}.$$

The following theorem asserts that the sequence(11) is monotone.

Theorem 5.1 *Consider the optimization problem (1)*

$$\inf_{(x,u)\in H\times H} f(x, u) = f^{inf} \tag{1}$$

where $f(x, u)$ is a convex function on $H \times H$ and the iterative procedure (11). *Suppose there exist an ε-neighborhood , X_{f_ε} of the set X_f for which*

$$\big\langle(\eta, \xi), (x, u) - (y, v)\big\rangle \neq 0, \quad (\eta, \xi) \in \partial f(x, u),$$

$$(y, v) \in Pr_{X_f}(x, u), \quad (x, u) \in X_{f_\varepsilon} \setminus X_f$$

and

$$\liminf_{k\to\infty} \|m(\partial f(x_k, u_k))\| \neq 0,$$

where inf *is taken at all sequences $(x_k, u_k) \in X_{f_\varepsilon} \setminus X_f$. We recall that*

$$\big\langle(v, y), (x, u)\big\rangle = \langle v, x\rangle + \langle u, y\rangle$$

for all x and y in H and $u(x)$ and $v(y)$ in H.

If $(x_0, u_0) \in X_{f_\varepsilon}$ and the following inequality is satisfied:

$$f(x_k, u_k) - \langle m(\mathbf{0}, \partial f_k), (x_k, u_k) - (y_k, v_k) \rangle < \varepsilon^k, \tag{12}$$

where $(y_k, v_k) \in Pr_{X_f}(x_k, u_k)$ and $\varepsilon^k < \frac{q}{2} f(x_k, u_k)$, $0 < q < 1$, then the following inequality:

$$\max \|(x_{k+1}, u_{k+1}) - Pr_{X_f}(x_{k+1}, u_{k+1})\| \leq \min \|(x_k, u_k) - Pr_{X_f}(x_k, u_k)\| \tag{13}$$

is valid for every $k = 0, 1, 2, \ldots$.

On the basis of inequalities (12) and (13) we obtain the following inequalities:

$$\|T_f(x_k)\|^2 - \langle m(\mathbf{0}, T_{f\,k}), (x_k - y_k) \rangle < \varepsilon_1^k, \tag{14}$$

where $(y_k, v_k) \in Pr_{X_f}(x_k, u_k)$ and $\varepsilon_1^k < \frac{q}{2} \|T_f(x_k)\|^2$, $0 < q < 1$, and finally:

$$\max \|x_{k+1} - y_{k+1}\| \leq \min \|x_k - y_k\| \tag{15}$$

for every $k = 0, 1, 2, \ldots$.

And therefore the iterative procedure (10) can be replaced with the iterative procedure (11).

Proof. As long as $f(x, u)$ is convex under (13), for any $(y_k, v_k) \in Pr_{X_f}(x_k, u_k)$, $k = 0, 1, 2, \ldots$, we obtain

$$0 \leq A_k = \frac{\|x_{k+1} - y_{k+1}\|^2}{\|x_k - y_k\|^2} \leq$$

$$\frac{\|(x_{k+1}, u_{k+1}) - (y_{k+1}, v_{k+1})\|^2}{\|(x_k, u_k) - (y_k, v_k)\|^2} =$$

$$\frac{\|(x_{k+1},u_{k+1})-(y_{k+1},v_{k+1})\|^2}{\|(x_k,u_k)-(y_k,v_k)\|^2} \leq \frac{\|(x_{k+1},u_{k+1})-(y_k,v_k)\|^2}{\|(x_k,u_k)-(y_k,v_k)\|^2} =$$

$$\frac{1}{\|(x_k,u_k)-(y_k,v_k)\|^2}\|(y_k,v_k)-(x_k,u_k)+\frac{f(x_k,u_k)m(\mathbf{0},\partial f_k)}{\|m(\mathbf{0},\partial f_k)\|^2}\|^2 =$$

$$\frac{(f(x_k,u_k))^2}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2}+1+2\frac{f(x_k,u_k)\langle m(\mathbf{0},\partial f_k),(y_k,v_k)-(x_k,u_k)\rangle}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2}$$

Under the inequality $\varepsilon^k \leq \frac{q}{2}F(x_k,u_k)$ and (12), we complete the proof:

$$A_k \leq \frac{(f(x_k,u_k))^2}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2}+1+2\frac{(\varepsilon^k-f(x_k,u_k))f(x_k,u_k)}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2} =$$

$$1-\frac{(f(x_k,u_k))^2}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2}+2\frac{\varepsilon^k f(x_k,u_k)}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2} \leq$$

$$1-(1-q)\frac{(f(x_k,u_k))^2}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2} = 1-(1-q)(B_k)^2 < 1,$$

where

$$B_k = \frac{(f(x_k,u_k))^2}{\|m(\mathbf{0},\partial f_k)\|^2\|(x_k,u_k)-(y_k,v_k)\|^2}$$

Thus, for every $(x_k,u_k) \in X_{f_\varepsilon} \setminus X_f$, $k=1,2,\dots$ we obtain

$$0 \leq A_k < 1-(1-q)(B_k)^2 < 1.$$

hence there exists $\varepsilon_1^k \leq \varepsilon^k$ such that

$$max\,\|x_{k+1}-y_{k+1}\| \leq \min\,\|x_k-y_k\| \tag{15}$$

□

6 Numerical Examples

In this section we apply the introduced idea of simplification of the solution of convex unconstrained continuos differentiable optimization problem to create an iterative procedure of one independent parameter for solving the optimization problem. We show that these new iterative procedures solve the convex optimization problems faster with same precision than the existing optimization procedures.

We consider two real valued convex functions $f(x, y)$, supposing $x \in R^1$ and $y(x)$ is from the dual space with values in R^1.

The dual space of the Euclidean space R^1 of real numbers, which is also a Hilbert space, is also one dimentional linear space. We remind that R^1 can be considered also as a vector space with only two possible directions, positive and negative.

We consider two simple examples of finding the minimum of convex functions of two variables, supposing one of them is a function of the other one, usually y is a map from R^1 to R^1, or i.e. $y : R \to R$.

Because of the convexity, the optimization procedures with two variables can be simplified to procedures with one variable, which is a simple presentation of the idea of simplification of the optimization process introduced above.

As shown in the previous section for solving numerically an optimization problem instead of using the next iterative procedure:

$$(x_{k+1}, u_{k+1}) = (x_k, u_k) - \frac{f(x_k, u_k)m(\mathbf{0}, \partial f_k)}{\|m(\mathbf{0}, \partial f_k)\|^2}, \ k = 0, 1, \ldots. \tag{10}$$

we can solve it with the following simpler iterative procedure:

$$x_{k+1} = x_k - \frac{\|T_f(x_k)\|^2 m(\mathbf{0}, T_{f\,k})}{\|m(\mathbf{0}, T_{f\,k})\|^2},\ u_k = T_f(x_k),\ k = 0, 1, \ldots. \tag{11}$$

We present next two simple examples.

Numerical Example 1.

Let us solve the unconstrained convex optimization problem (see Fig. 1):

$$\min f_p(x, y),$$

where $f_p(x, y) = 4(x + p)^2 + 4y^2 - 4(x + p)y - 12y$, with initial point $(x, y) = (5, 5)$ and with precision $\varepsilon = 0.000001.p$ is a parameter of disturbances.

The partial derivatives should be equal to zero at the $\min f_p(x, y)$

$$f_{p_x}(x, y) = 8(x + p) - 4y = 0 \qquad \text{and} \qquad f_{p_y}(x, y) = 8y - 4(x + p) - 12 = 0.$$

We have to find the conditions for the parameter of distubances such that to have a monotone convergence algorithm.

Because both partial derivatives should be equal to zero at the solution point, then we have

$$y_p = 2(x + p) \quad \text{and} \quad y_p = \frac{x + p + 3}{2} \tag{$*$}$$

They are monotone, because $f_p(x, y)$ is convex.

These two lines intersect at different solution points for a given value of the parameter p and we use a sequence $\{p_k\}$ to create an iterative procedure depending only on x_{p_k} and the parameter of disturbances

p_{k+1} on each step. As the solution is a unique one when $p = 0$ we can find sufficient conditions for the parameter of disturbances p to ensure convergence.

This simplification is an analog of the monotone operator T_f.

We create an iterative procedute for $x_{p_{k+1}}$ for finding the minimum of $f_p(x, y)$:

$$x_{p_{k+1}} = x_{p_k} - \frac{\frac{9}{4}(x_{p_k} + p_{k+1} - 1)^2}{\frac{9}{2}(x_{p_k} + p_{k+1} - 1)} = x_{p_k} - \frac{x_{p_k} + p_{k+1} - 1}{2}$$

To ensure convergence we have to have

$$\max \|(x_{p_k}, y_{p_k}) - (x_{p_{k+1}}, y_{p_{k+1}}\| < \varepsilon_k,$$

where the sequence $\{\varepsilon_k\}$ is convergent to zero, i.e. the $\liminf_{k\to\infty} \varepsilon_k = 0^+$, $\varepsilon_k > 0$ for every positive integer k. Because

$$\max |x_{p_k} - x_{p_{k+1}}| \leq \max \|(x_{p_k}, y_{p_k}) - (x_{p_{k+1}}, y_{p_{k+1}}\| < \varepsilon_k,$$

means that

$$|x_{p_k} - x_{p_{k+1}}| = |x_{p_k} - 1 + p_{k+1}| < 2\varepsilon_k.$$

We receive

$$|x_{p_k} - 1 + p_{k+1}| \leq |x_{p_k} - 1| + |p_{k+1}| < 2\varepsilon_k^1$$

where the sequence $\{\varepsilon_k^1\}$ is also convergent to zero, i.e. the $\liminf_{k\to\infty} \varepsilon_k^1 = 0^+$, $\varepsilon_k^1 \geq \varepsilon_k > 0$ for every positive integer k, such that when $|x_{p_k} - 1| + |p_{k+1}| < 2\varepsilon_k^1$ then $|x_{p_k} - 1 + p_{k+1}| < 2\varepsilon_k$, or

$$0 \leq |p_{k+1}| < 2\varepsilon_k^1 - |x_{p_k} - 1|$$

or

$$|x_{p_k} - 1| - 2\varepsilon_k^1 \leq p_{k+1} \leq 2\varepsilon_k^1 - |x_{p_k} - 1|$$

From equations $(*)$ it follows that when p_k approaches 0 and x_{p_k} approaches $x^* = 1$ then y_{p_k} approaches $y^* = 2$.

We can simplify the iterative procedure by presenting the disturbances on each step k for any $x_k \in \{x_{p_k}\}$ with addition a one parameter p_{k+1}:

$$x_{p_{k+1}} = x_k - \frac{x_k - 1}{2} + p_{k+1}$$

To have for any $x_{k+1} \in \{x_{p_{k+1}}\}$

$$\lim_{k\to\infty} |x_k - x_{k+1}| \to 0$$

p_{k+1} should satisfy next inequalities:

$$-\left|\frac{x_k - 1}{2}\right| < p_{k+1} < \left|\frac{x_k - 1}{2}\right|$$

for every $x_k \in \{x_{p_k}\}$.

Of course, here the direct (analytical) solution is easier. In Example 2 is not so.

We compare our results with the results of well known modifications of Newton's method using MATLAB.

a) Solving the problem using our algorithm, we received next results: $x = 1.0000$ and $y = 2.0000$ for time $t = 1.8896 \times 10^{-4}$ sec.

b) Solving the problem using the algorithm FMINUNC from MATLAB, we received next results: $x =$

1.0000 and $y = 2.0000$ for time $t = 0.0089$ sec.

c) Solving the problem using the algorithm FMINSEARCH from MATLAB, we received next results:

$x = 1.0000$ and $y = 2.0000$ for time $t = 0.0026$ sec.

These results are summarized in the next table.

Algorithm	x - results	y - results	time
Our Algorithm	1.0000	2.0000	1.8896×10^{-4} sec.
FMINUNC	1.0000	2.0000	0.0089 sec.
FMINSEARCH	1.0000	2.0000	0.0026 sec.

The advantages of our algorithm are obvious.

$$\mathbf{f(x,y)} = 4x^2 + 4y^2 - 4xy - 12y$$

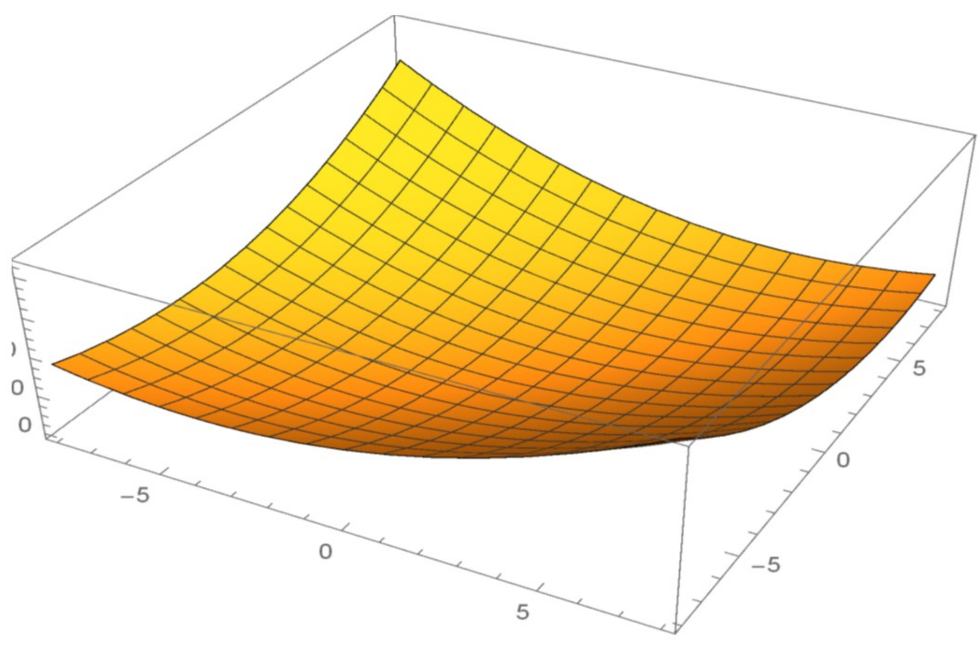

Figure 1.

Numerical Example 2.

We consider the following unconstrained convex optimization problem in which the surface is flatter close to the solution point (see Fig. 2.):

$$\min f_p(x,y),$$

where $f_p(x,y) = (x+p)^6 + e^{(x+p)} + (x+p)^2 + 2(x+p)y + y^2 + y^4$, with the same initial point $(x,y) = (5,5)$ and with the same precision $\varepsilon = 0.000001$. p is again a parameter of disturbances.

The partial derivatives should be equal to zero at the $\min f_p(x,y)$

$$f_{p_x}(x,y) = 6(x+p)^5 + e^{(x+p)} + 2(x+p) + 2y = 0 \qquad \text{and} \qquad f_{p_y}(x,y) = 4y^3 + 2y + 2(x+p) = 0.$$

They are monotone, because $f_p(x,y)$ is convex.

Because both partial derivatives should be equal to zero at the solution point, then we have

$$y_p = -\frac{6(x+p)^5 + e^{(x+p)} + 2(x+p)}{2} \quad \text{and} \quad y_p = \Big(\frac{6(x+p)^5 + e^{(x+p)}}{4}\Big)^{1/3} \qquad (**)$$

These two lines intersect at different solution points for a given value of the parameter p and we use a sequence $\{p_k\}$ to create an iterative procedure depending only on x_{p_k} and the parameter of disturbances p_{k+1} on each step. As the solution is a unique one when $p = 0$ we can find sufficient conditions for the parameter of disturbances p to ensure convergence.

This simplification is an analog of the monotone operator T_f.

We create an iterative procedute for $x_{p_{k+1}}$ for finding the minimum of $f_p(x,y)$:

$$F_{p_{k+1}} = \Big\{\Big(\frac{6(x_{p_k} + p_{k+1})^5 + e^{(x_{p_k}+p_{k+1})}}{4}\Big)^{1/3} + \frac{6(x_{p_k} + p_{k+1})^5 + 2(x_{p_k} + p_{k+1}) + e^{(x_{p_k}+p_{k+1})}}{2}\Big\}^2,$$

$$G_{p_{k+1}} = 2\Big\{\Big(\frac{6(x_{p_k}+p_{k+1})^5+e^{(x_{p_k}+p_{k+1})}}{4}\Big)^{1/3} + \frac{6(x_{p_k}+p_{k+1})^5+2(x_{p_k}+p_{k+1})+e^{(x_{p_k}+p_{k+1})}}{2}\Big\}, \text{and}$$

$$H_{p_{k+1}} = \Big\{\Big(\frac{1}{12}\Big)\Big[\frac{4}{6(x_{p_k}+p_{k+1})^5+e^{(x_{p_k}+p_{k+1}}}\Big]^{2/3}(30(x_{p_k}+p_{k+1})^4+e^{(x_{p_k}+p_{k+1})})+0.5(30(x_{p_k}+p_{k+1})^4+e^{(x_{p_k}+p_{k+1})}+2)\Big\}$$

Then

$$x_{p_{k+1}} = x_{p_k} - \frac{F_{p_{k+1}}}{G_{p_{k+1}}H_{p_{k+1}}}$$

To ensure convergence we have to have

$$\max \|(x_{p_k}, y_{p_k}) - (x_{p_{k+1}}, y_{p_{k+1}}\| < \varepsilon_k,$$

where the sequence $\{\varepsilon_k\}$ is convergent to zero, i.e. the $\liminf_{k\to\infty} \varepsilon_k = 0^+$, $\varepsilon_k > 0$ for every positive integer k. Because

$$\max |x_{p_k} - x_{p_{k+1}}| \leq \max \|(x_{p_k}, y_{p_k}) - (x_{p_{k+1}}, y_{p_{k+1}}\| < \varepsilon_k,$$

means that

$$|x_{p_k} - x_{p_{k+1}}| = \frac{F_{p_{k+1}}}{|G_{p_{k+1}}|H_{p_{k+1}}} < \varepsilon_k,$$

since $F_{p_{k+1}} \geq 0$ and $H_{p_{k+1}} \geq 0$ for every p_{k+1} satisfying the convergnce conditions.

Therefore we have to restrict p_{k+1} such that, when $|p_{k+1}| < \varepsilon^1_{k+1}$, then $\frac{F_{p_k}}{|G_{p_k}|H_{p_k}} < \varepsilon_k$. Here $\{\varepsilon^1_k\}$ is a decreasing sequence with positive elemets that ensures these requirements.

From equations $(**)$ it follows that when p_k approaches 0 and x_{p_k} approaches $x^* = 5$ then y_{p_k} approaches $y^* = 5$.

We can simplify the iterative procedure by presenting the disturbances on each step k for any $x_k \in \{x_{p_k}\}$ with addition a one parameter p_{k+1}.

We use the following simplified notarions

$$F_k = \Big\{\Big(\frac{6x_k^5 + e^{x_k}}{4}\Big)^{1/3} + \frac{6x_k^5 + 2x_k + e^{x_k}}{2}\Big\}^2,$$

$$G_k = 2\Big\{\Big(\frac{6x_k^5 + e^{x_k}}{4}\Big)^{1/3} + \frac{6x_k^5 + 2x_k + e^{x_k}}{2}\Big\}, \text{and}$$

$$H_k = \Big\{\Big(\frac{1}{12}\Big)\Big[\frac{4}{6x_k^5 + e^{x_k}}\Big]^{2/3}(30x_k^4 + e^{x_k}) + 0.5(30x_k^4 + e^{x_k} + 2)\Big\}.$$

and receive the iterative procedure:

$$x_{p_{k+1}} = x_k - \frac{F_k}{G_k H_k} + p_{k+1}$$

To have for any $x_k \in \{x_{p_k}\}$ and $x_{k+1} \in \{x_{p_{k+1}}\}$

$$\lim_{k\to\infty} |x_k - x_{k+1}| \to 0$$

where

$$|x_k - x_{k+1}| = \Big|\frac{F_k}{|G_k| H_k} - p_{k+1}\Big|$$

p_{k+1} should satisfy next inequalities:

$$-\frac{F_k}{|G_k| H_k} < p_{k+1} < \frac{F_k}{|G_k| H_k}$$

The direct analytical solution is not so easy here as in Example 1.

a) Solving the problem using our algorithm, we received next results: $x = -0.5493$ and $y = 0.4107$ for time $t = 3.6133 \times 10^{-4}$ sec.

b) Solving the problem using the algorithm FMINUNC from MATLAB, we received next results: $x = -0.5487$ and $y = 0.4117$ for time $t = 0.0110$ sec.

c) Solving the problem using the algorithm FMINSEARCH from MATLAB, we received next results:

$x = -0.5493$ and $y = 0.4107$ for time $t = 0.0034$ sec.

These results are summarized in the next table.

Algorithm	x - results	y - results	time
Our Algorithm	-0.5493	0.4107	3.6133 $\times 10^{-4}$ sec.
FMINUNC	-0.5487	0.4117	0.0110 sec.
FMINSEARCH	-0.5493	0.4107	0.0034 sec.

The advantages of our algorithm are clear again.

$$\mathbf{f(x,y)} = x^6 + e^x + x^2 + 2xy + y^2 + y^4$$

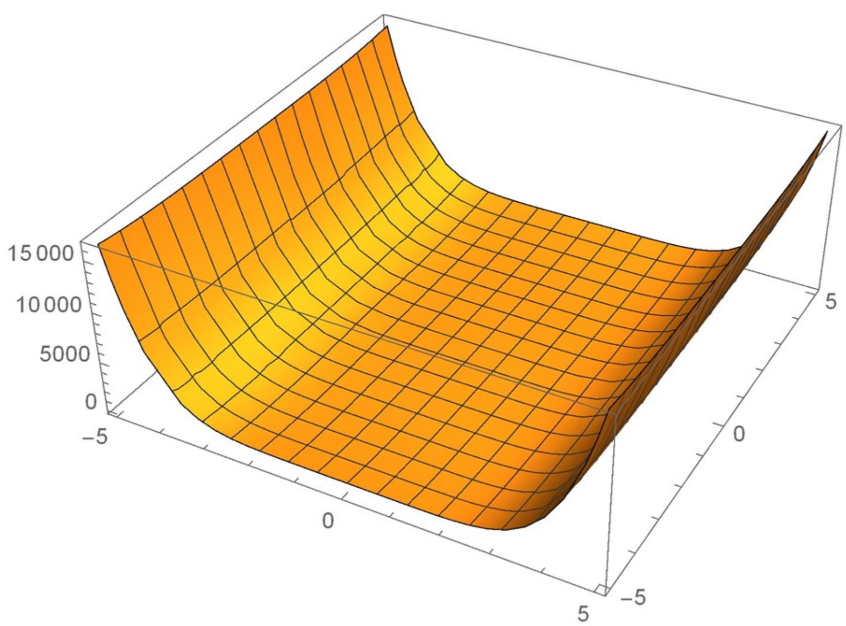

Figure 2.

7 A New Example of a Fitzpatrick Function

We considered in Theorem 3.3 the optimization problem

$$\inf_{(x,u)\in H\times H} f(x,u). \tag{1}$$

and proved that to solve this problem we could substitute the solution of differential inclusion

$$(\dot{x},\dot{u}) \in -\partial f(x(t),u(t)) \tag{3}$$

$$(x(0),u(0)) = (x_0,u_0).$$

by solving differential inclusion

$$(\dot{x},\dot{u}) \in -\frac{\partial f(x(t),u(t))}{\|m(\mathbf{0},\partial f(t))\|^2}, \tag{6}$$

$$(x(0),u(0)) = (x_0,u_0),$$

We showed that the solutions of differential inclusions (3) and (6) coincide.

We proved there also that $f(x,u)$ is a Lyapunov function for the differential inclusion (6). Therefore this is an example that the Fitzparick function appears to be a Lyapunov fuction.

We showed in Theorem 5.1 that to solve the problem numerically we can substitute its corresponding iterative procedure

$$(x_{k+1},u_{k+1}) = (x_k,u_k) - \frac{f(x_k,u_k)m(\mathbf{0},\partial f_k)}{\|m(\mathbf{0},\partial f_k)\|^2},\ k=0,1,\ldots. \tag{10}$$

which is based on the differential inclusion,

$$(\dot{x},\dot{u}) \in -\frac{\partial L_T(x(t),u(t))}{\|m(\mathbf{0},\partial L_T(t))\|^2}, \tag{6}$$

$$(x(0),u(0))=(x_0,u_0)$$

with the iterative procedure

$$x_{k+1} = x_k - \frac{\|T_f(x_k)\|^2 m(\mathbf{0},T_{f\,k})}{\|m(\mathbf{0},T_{f\,k})\|^2},\ u_k = T_f(x_k),\ k = 0,1,\dots. \tag{11}$$

based on differential inclusion:

$$\dot{x} \in -T_f(x(t)), x(0) = x_0 \tag{2}$$

A Lyapunov function for the differential inclusion (9) is $(\|T_f(x)\|)^2$

Ivanov and Raykov in [12] developed a parametric Lyapunov function method for solving nonlinear systems in Hilbert space. We show close connections between the Lyapunov function created to solve the given problem and Fitzpatrick functions.

Let $f_i(x,u), i = 1,2,\dots,m$ be nonsmooth scalar functions which are defined on $H \times H$ where $\mathbf{H}$ is a Hilbert space and let us consider the following system of equations

$$f_i(x,u) = 0, \quad i = 1,2,\dots,m. \tag{16}$$

We construct a depending on $p(\varepsilon)$ family of functions:

$$V_{p(\varepsilon)}(x,u) = \sum_{i=1}^{m} \alpha_i |f_i(x,x^*)|^{1+p_i(\varepsilon)} \geq 0, \tag{17}$$

where $\alpha_i > 0$, and $p(\varepsilon) = (p_1(\varepsilon), p_2(\varepsilon), \ldots, p_m(\varepsilon))$ is a vector parameter such that $p_i(\varepsilon) \in (p_i - \varepsilon, p_i + \varepsilon)$ and $p_i - \varepsilon > -1, \varepsilon > 0, \quad i = 1, 2, \ldots, m.$

According to [10] the generalized directional derivative is

$$f^0((x,u);(t,s)) = \limsup_{(y,v)\Longrightarrow(x,u),\, \lambda\downarrow 0} \frac{f((y,v)+\lambda(t,s)) - f(y,v)}{\lambda}$$

and the generalized gradient is

$$\partial f(x,u) = \{(\eta,\zeta) \in \mathbf{H}\times\mathbf{H} \,|\, f^0((x,u);(t,s)) \geq \big\langle(\eta,\zeta),(t,s)\big\rangle \,\forall\,(t,s) \in \mathbf{H}\times\mathbf{H}\}$$

Definition 7.1 ([10]) *f is said to be* regular *at x provided*

(i) *For all v, the usual one-side directional derivative $f'(x;v)$ exists.*

(ii) *For all v, $f'(x;v) = f^0(x;v)$.*

We consider functions $f_i(x,u)$, $i = 1, 2, \ldots, m$ that are regular and assume that $V_{p(\varepsilon)}(x,u)$ is a convex function. Note that the convex function is regular.

We denote by $m(\mathbf{0}, \partial V_{p(\varepsilon)}(x,u)$ the metric projection of the origin to the set $\partial V_{p(\varepsilon)}(x,u)$.

Let $0 \notin \partial V_{p(\varepsilon)}(x_0,u_0)$. Consider the following differential inclusion:

$$(\dot{x},\dot{u}) \in -\frac{\partial V_{p(\varepsilon)}(x,u)}{\|m(\mathbf{0}, \partial V_{p(\varepsilon)}(x,u))\|^2}, \quad (x(0),u(0)) = (x_0,u_0). \tag{18}$$

As long as $f_i(x)$ are regular functions, according to [8] we have

$$\partial V_{p(\varepsilon)}(x,u) = \sum_{i=1}^{m} \alpha_i \|f_i(x,u)\|^{p_i(\varepsilon)} (\operatorname{sign} f_i(x,u)) \partial f_i(x,u)(1 + p_i(\varepsilon)). \tag{19}$$

The differential inclusion (18) is the Filippov extension of the following differential equation with possibly discontinuous right-hand side, see [8] and [11]:

$$(\dot{x}, \dot{u}) = -\frac{m(\partial V_{p(\varepsilon)}(x,u))}{\|m(\mathbf{0}, \partial V_{p(\varepsilon)}(x,u))\|^2}, \quad (x(0), u(0)) = (x_0, u_0) \tag{20}$$

As long as $V_{p(\varepsilon)}(x)$ is a convex function, the subdifferential $\partial V_{p(\varepsilon)}(x)$ is a maximal monotone operator and the differential inclusion

$$(\dot{x}, \dot{u}) \in -\partial V_{p(\varepsilon)}(x,u), \quad (x(0), u(0)) = (x_0, u_0) \tag{25}$$

has an unique solution, see [8] .

We see that so created Lyapunov function $V_{p(\varepsilon)}(x,u)$ is also an example of a Fitzpatrick function.

Using this method, the number of the equations is independent from the number of independent variables.

We could consider at the end a simplification of problem

$$\inf_{(x,u)\in H\times H} f(x,u). \tag{1}$$

$T_f x$ is from Definition 3.1.

We can use differential inclusion

$$\dot{x} \in -T_f(x(t)), x(0) = x_0 \tag{9}$$

to solve the problem with a Lyapunov function $(\|T_f x\|)^2$.

8 Concluding Remarks

Our interest in optimizing Fitzpatrick functions is motivated by the following considerations:

1. Optimizing Fitzpatrick functions can be simplified from optimization problems on $H \times H$ to optimization problems on H. It would be of interest to explore classes of convex functions on $H \times H$ which have a similar simplification property.

2. We introduce an iterative procedure for solving problems of this type. Because of the convexity the procedure can be simplified.

3. We present two examples of Fitzpatrick functions which are also Lyapunov functions. This may lead to further connections with Lyapunov functions.

4. Fitzpatrick functions are convex on $H \times H$ and involve implicit functions.([13]

5. We suppose this type of functions could be used for solving a larger scale of nonconvex optimization problems (see [14]).

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