General Factoring of Binomials of the form $a^n \pm b^n$

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Abstract

This paper explores the complete factoring of $a^n \pm b^n$ for any value of n. It contains heretofore-unpublished details on the method of factoring such binomials, including finding factors that are not immediately obvious. The key to the complete factoring is the prime factorization of the exponent n. Beyond the first direct factors are a number of indirectly determined factors that are the results of many polynomial divisions that were needed to find them. The methodology required to identify those factors is described. This ultimately leads to a formulation, based on binomial coefficients, for the total number of factors of such polynomials. The work contains a detailed factoring of several binomials, especially $a^{105} + b^{105}$ and $a^{252} - b^{252}$, which demonstrate everything presented in the paper. The complete factorization of $a^{105} + b^{105}$ is given in detail to demonstrate the methodology of the procedure for finding the indirectly determined factors and its effectiveness. The complete factorization of $a^{252} - b^{252}$ is given in detail to demonstrate all of the various formulas involved in the complete factoring. In the end, binomial coefficients are shown to be the key to identifying the number of factors leading to a formula that predicts the total number of factors of such a binomial.

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This paper presents a theory and practice for the complete factoring $a^n \pm b^n$ for any n. A literature search on this subject only turned up one relevant paper, but some others have been listed in the list of references. The one relevant paper is an article on "Factoring" by N. Barrows of Kimball Union Academy, (Barrows, 170). In it Barrows shows the factoring of $a^2 - b^2$ into $(a - b)\bullet(a + b)$ for various a and b. Barrows also shows that $a^n \pm b^n$ can be factored into $(a \pm b)$ times a longer expression of degree n-1. This is the only article I could find on factoring beyond the customary factoring of $a^2 - b^2$, $a^3 - b^3$, & $a^3 + b^3$. However, the inspiration for part of this paper came from a small article by Andy Parker, "Discovering a New Factor Form" in *Mathematics Teacher*, (Parker, 215).

The key to the factoring of $a^n \pm b^n$ is the prime factoring of $n = 2^p\prod_{i=1}^{m} f_i$. For each of the m odd prime factors f_i there will be a factor of $a^n \pm b^n$ of f_i terms.

The general forms of the factors corresponding to the odd prime factor k of n, where $n = jk$, are: **(These formulas will be referred to as formula I.)**

$a^{jk} - b^{jk} = (a^j - b^j)\bullet(a^{(k-1)j} + a^{(k-2)j}b^j + a^{(k-3)j}b^{2j} + a^{(k-4)j}b^{3j} + \ldots + b^{(k-1)j})$

or $= (a^j - b^j)\bullet(\sum_{i=0}^{k-1} a^{(k-1-i)j}b^{ij})$.

and $a^{jk} + b^{jk} = (a^j + b^j)\bullet(a^{(k-1)j} - a^{(k-2)j}b^j + a^{(k-3)j}b^{2j} - a^{(k-4)j}b^{3j} + \ldots + b^{(k-1)j})$

or $= (a^j + b^j)\bullet(\sum_{i=0}^{k-1} (-1)^i a^{(k-1-i)j}b^{ij})$.

The factors based on the 2^p prime factors of n will all be binomials.
The factors of $a^c - b^c$, where $c=2^p$, will be $(a - b)$ and p factors of the form $a^c + b^c$, where $c=2^i$ for i=0…p-1. **(Formula II.)**
For example, the factors of $a^8 - b^8 = (a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^4 + b^4)$
For $a^c + b^c$, where $c=2^p$, there is no other factor.

Any factor of the form $\sum_{i=0}^{k} a^{2i}b^{2(k-i)}$ can also be factored: $\sum_{i=0}^{k} a^i b^{(k-i)} \bullet \sum_{i=0}^{k} (-1)^i a^i b^{(k-i)}$

(This is formula III.)

For example: consider factoring $a^{10} + b^{10}$ and $a^{10} - b^{10}$, 10=2•5, m=1, p=1, j=2, k=5

$a^{10} + b^{10} = (a^2 + b^2)\bullet(a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8)$,

$a^{10} - b^{10} = (a - b)\bullet(a + b)\bullet(a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8)$

$= (a - b)\bullet(a + b)\bullet(a^4 + a^3b + a^2b^2 + ab^3 + b^4)\bullet(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

and $a^{112} - b^{112} = (a^{16}-b^{16})\bullet(a^{96}+a^{80}b^{16}+a^{64}b^{32}+a^{48}b^{48}+a^{32}b^{64}+a^{16}b^{80}+b^{96})$ [$112=2^4\bullet7$]

$= (a-b)\bullet(a+b)\bullet(a^2+b^2)\bullet(a^4+b^4)\bullet(a^8+b^8)$

$\bullet(a^{48}-a^{40}b^8+a^{32}b^{16}-a^{24}b^{24}+a^{16}b^{32}-a^8b^{40}+b^{48})\bullet(a^{24}-a^{20}b^4+a^{16}b^8-a^{12}b^{12}+a^8b^{16}-a^4b^{20}+b^{24})\bullet(a^{12}-a^{10}b^2+a^8b^4-a^6b^6+a^4b^8-a^2b^{10}+b^{12})$

$\bullet(a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6)\bullet(a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6)$

$(a^{96} + a^{80}b^{16} + a^{64}b^{32} + a^{48}b^{48} + a^{32}b^{64} + a^{16}b^{80} + b^{96})$ is factored four times (p = 4).

In the case of $a^{60} + b^{60}$: $60 = 2^2 \bullet 3 \bullet 5$, m=2, p=2. For the 5-term factor based on the prime factor 5 of n, k=5, $j=2^2 \bullet 3=12$. The first binary factor taken out will be $(a^{12} + b^{12})$ and the exponents of the five terms in the other factor will be multiples of 12.

$$a^{60} + b^{60} = (a^{12}+b^{12})\bullet(a^{48} - a^{36}b^{12} + a^{24}b^{24} - a^{12}b^{36} + b^{48})$$

Then $(a^{12} + b^{12})$ will be factored as $jk = 4\bullet3$, k=3, j=4: $(a^4 + b^4)\bullet(a^8 - a^4b^4 + b^8)$

$$a^{60} + b^{60} = (a^4 + b^4)\bullet(a^8 - a^4b^4 + b^8)\bullet(a^{48} - a^{36}b^{12} + a^{24}b^{24} - a^{12}b^{36} + b^{48})$$

In this case, m+1 = 3 giving us the three factors with 2, 3, & 5 terms.

Similarly,

$$a^{60} - b^{60} = (a^4 - b^4)\bullet(a^8 + a^4b^4 + b^8)\bullet(a^{48} + a^{36}b^{12} + a^{24}b^{24} + a^{12}b^{36} + b^{48})$$

$$= (a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^8 + a^4b^4 + b^8)\bullet(a^{48} + a^{36}b^{12} + a^{24}b^{24} + a^{12}b^{36} + b^{48})$$

$$= (a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^4 + a^2b^2+ b^4)\bullet(a^4 - a^2b^2 + b^4)\bullet(a^{48} + a^{36}b^{12} + a^{24}b^{24} + a^{12}b^{36} + b^{48})$$

$$= (a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^2 + ab+ b^2)\bullet(a^2 - ab + b^2)\bullet(a^4 - a^2b^2 + b^4)\bullet(a^{48} + a^{36}b^{12} + a^{24}b^{24} + a^{12}b^{36} + b^{48})$$

$$= (a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^2 + ab+ b^2)\bullet(a^2 - ab + b^2)\bullet(a^4 - a^2b^2 + b^4)\bullet(a^{24} + a^{18}b^6 + a^{12}b^{12} + a^6b^{18} + b^{24})\bullet(a^{24} - a^{18}b^6 + a^{12}b^{12} - a^6b^{18} + b^{24})$$

$$= (a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^2 + ab+ b^2)\bullet(a^2 - ab + b^2)\bullet(a^4 - a^2b^2 + b^4)\bullet(a^{12} + a^9b^3 + a^6b^6 + a^3b^9 + b^{12})\bullet(a^{12} - a^9b^3 + a^6b^6 - a^3b^9 + b^{12})\bullet(a^{24} - a^{18}b^6 + a^{12}b^{12} - a^6b^{18} + b^{24})$$

The factors arising from 2^p should be postponed to the last; otherwise, it will be necessary to factor a large expression by grouping to get the smaller factors. As an example of the reason for postponing the binary factors based on 2^p, consider factoring $(a^9 + b^9)$. $n=3\bullet3$, m=2 and p=0. There should be 2+1 = 3 factors, a binary and two with 3 terms each. But if we start with the factor $(a + b)$, we get a factor of nine terms.

$$a^9 + b^9 = (a + b)\bullet(a^8 - a^7b + a^6b^2 - a^5b^3 + a^4b^4 - a^3b^5 + a^2b^6 - ab^7 + b^8)$$

The large factor can be further factored by grouping:

$$(a^8 - a^7b + a^6b^2 - a^5b^3 + a^4b^4 - a^3b^5 + a^2b^6 - ab^7 + b^8)$$
$$=(a^6(a^2 - ab + b^2) - a^3b^3(a^2 - ab + b^2) + b^6(a^2 - ab + b^2))$$
$$=(a^2 - ab + b^2)\bullet(a^6 - a^3b^3 + b^6)$$
$$a^9 + b^9 =(a + b)\bullet(a^2 - ab + b^2)\bullet(a^6 - a^3b^3 + b^6)$$

Which is the same result as when we start with the prime factor 3:

$$a^9 + b^9 = (a^3 + b^3)\bullet(a^6 - a^3b^3 + b^6)$$
$$= (a + b)\bullet(a^2 - ab + b^2)\bullet(a^6 - a^3b^3 + b^6)$$

The general form of these factors is illustrated by the following **(Formula IV.).** n is a product ($n=efg$ in this case) and the factor to be factored by grouping contains efg terms**:**

$$a^{efg} + b^{efg} = (a+b) \bullet(a^{efg-1} - a^{efg-2}b + a^{efg-3}b^2 - a^{efg-3}b^3 + \dots + b^{efg-1)})$$
$$= (a+b) \bullet (a^{e-1} - a^{e-2}b + a^{e-3}b^2 - a^{e-3}b^3 + \dots + b^{e-1})$$
$$\bullet(a^{e(f-1)} - a^{e(f-2)}b^e + a^{e(f-3)}b^{2e} - a^{e(f-3)}b^{3e} + \dots + b^{e(f-1)})$$
$$\bullet(a^{ef(g-1)} - a^{ef(g-2)}b^{ef} + a^{ef(g-3)}b^{2ef} - a^{ef(g-3)}b^{3ef} + \dots + b^{ef(g-1)})$$

$a^{efg} - b^{efg}$ is similar, but all the signs in the factors after $(a-b)$ are plus signs.

When there are h instances of the same odd prime factor, f, one could use formula I with $k=f^h$, and then factoring by grouping can be used to finding the resulting factors. For f^h, there will be h factors of f terms each after the binomial factor and they will be of degrees $(f–1)\bullet f^i$ for i=0 to h-1.

The preceding shows this for 3^2. Consider 5^2 and 3^3:

$$a^{25} – b^{25} = (a–b)\bullet(a^{24}+a^{23}b+a^{22}b^2+a^{21}b^3+a^{20}b^4+a^{19}b^5+a^{18}b^6+a^{17}b^7+a^{16}b^8$$
$$+a^{15}b^9+a^{14}b^{10}+a^{13}b^{11}+a^{12}b^{12}+a^{11}b^{13}+a^{10}b^{14}+a^9b^{15}+a^8b^{16}+a^7b^{17}$$
$$+a^6b^{18}+a^5b^{19}+a^4b^{20}+a^3b^{21}+a^2b^{22}+ab^{23}+b^{24})$$
$$= (a–b)\bullet(a^{20}(a^4+a^3b+a^2b^2+ab^3+b^4)+a^{15}b^5(a^4+a^3b+a^2b^2+ab^3$$
$$+b^4)+ a^{10}b^{10}(a^4+a^3b+a^2b^2+ab^3+b^4)+ a^5b^{15}(a^4+a^3b+a^2b^2$$
$$+ab^3+b^4)+ b^{20}(a^4+a^3b+a^2b^2+ab^3+b^4))$$
$$= (a–b) \bullet (a^4+a^3b+a^2b^2+ab^3+b^4) \bullet (a^{20}+a^{15}b^5+ a^{10}b^{10}+ a^5b^{15}+ b^{20})$$

$$a^{27} – b^{27} = (a–b)\bullet(a^{26}+a^{25}b+a^{24}b^2+a^{23}b^3+a^{22}b^4+a^{21}b^5+a^{20}b^6+a^{19}b^7+a^{18}b^8$$
$$+a^{17}b^9+a^{16}b^{10}+a^{15}b^{11}+a^{14}b^{12}+a^{13}b^{13}+a^{12}b^{14}+a^{11}b^{15}+a^{10}b^{16}+a^9b^{17}$$
$$+a^8b^{18}+a^7b^{19}+a^6b^{20}+a^5b^{21}+a^4b^{22}+a^3b^{23}+a^2b^{24}+ab^{25}+b^{26})$$
$$= (a–b) \bullet (a^{24}(a^2+ab+b^2)+ a^{21}b^3(a^2+ab+b^2)+ a^{18}b^6(a^2+ab+b^2)$$
$$+ a^{15}b^9(a^2+ab+b^2)+ a^{12}b^{12}(a^2+ab+b^2)+ a^9b^{15}(a^2b+ab+b^2)$$
$$+ a^6b^{18}(a^2+ab+b^2)+ a^3b^{21}(a^2+ab+b^2)+ b^{24}(a^2+ab+b^2))$$
$$= (a–b) \bullet (a^2+ab+b^2)$$
$$\bullet (a^{24}+ a^{21}b^3+ a^{18}b^6+ a^{15}b^9+ a^{12}b^{12}+ a^9b^{15}+ a^6b^{18}+ a^3b^{21}+ b^{24})$$
$$= (a–b) \bullet (a^2+ab+b^2)$$
$$\bullet (a^{18}(a^6+ a^3b^3+b^6)+ a^9b^9(a^6+ a^3b^3+ b^6)+ b^{18}(a^6+ a^3b^3+ b^6)$$
$$= (a–b) \bullet (a^2+ab+b^2) \bullet (a^6+ a^3b^3+ b^6) \bullet (a^{18}+ a^9b^9+ b^{18})$$

More Examples:

1. $a^5 + b^5 = (a + b)\bullet(a^4 – a^3b + a^2b^2 – ab^3 +b^4)$

   To see that this works, use an extension of FOIL giving:

   $a^5 – a^4b + a^3b^2 – a^2b^3 + ab^4 + a^4b – a^3b^2 + a^2b^3 – ab^4 + b^5$

   When combining terms, the middle terms cancel each other leaving $a^5 + b^5$

   The same goes for $a^5 – b^5$:

   $a^5 - b^5 = (a - b)\bullet(a^4 + a^3b + a^2b^3 + ab^4 +b^5)$

   The "-b" in the "a-b" factor causes all the terms resulting from the "–b" to be negative, canceling the corresponding ones from the "a" except for a^5 at the beginning and the -b^5 at the end.

2.

   Consider $a^4 – b^4$, m=0 and p=2, so there will be no factors with an odd number of terms, but there will be $(a – b)$ and two factors with exponents as powers of 2:

   $a^4 – b^4 = (a – b)\bullet(a + b)\bullet(a^2 + b^2)$

   Which is the same factoring we get from the factoring of $a^4 – b^4$ as a double instance of $a^2 – b^2$. I.e., $a^4 – b^4 = (a^2 – b^2)\bullet(a^2 + b^2) = (a – b)\bullet(a + b)\bullet(a^2 + b^2)$

   Note that for $a^4 + b^4$, m=0, p=2, and $a^4 + b^4$ is the only factor, there are no others.

3.

   $a^6 + b^6 = (a^2 + b^2) \bullet (a^4 - a^2b^2 + b^4)$ n=2•3, m=1, p=1, factor lengths 2 and 3

   $a^6 – b^6 = (a - b) \bullet (a + b) \bullet (a^4 + a^2b^2 + b^4)$, but the factors of $(a^4 + a^2b^2 + b^4)$ are $(a^2 + ab + b^2) \bullet (a^2 - ab + b^2)$

   $a^6 – b^6 = (a - b) \bullet (a + b) \bullet (a^2 + ab + b^2) \bullet (a^2 - ab + b^2)$

4.

$a^{20} - b^{20} = (a^4 - b^4)\bullet(a^{16} + a^{12}b^4 + a^8b^8 + a^4b^{12} + b^{16})$

$=(a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^{16} + a^{12}b^4 + a^8b^8 + a^4b^{12} + b^{16})$

$=(a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8)\bullet(a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8)$

$=(a - b)\bullet(a + b)\bullet(a^2 + b^2)\bullet(a^4 + a^3b + a^2b^2 + ab^3 + b^4)\bullet(a^4 - a^3b + a^2b^2 - ab^3 + b^8)\bullet(a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8)$

5.

Consider $a^{210} + b^{210}$, $210 = 2\bullet3\bullet5\bullet7$, $p=1$, $m=3$. There will be $m+1 = 4$ factors with 2, 3, 5, and 7 terms respectively. So for the first factoring $k=7$, $j=2\bullet3\bullet5 = 30$
$a^{210} + b^{210} = (a^{30} + b^{30})\bullet(a^{180} - a^{150}b^{30} + a^{120}b^{60} - a^{90}b^{90} + a^{60}b^{120} - a^{30}b^{150} + b^{180})$, 7 terms
Then to factor $(a^{30} + b^{30})$, $k=5$, $j=2\bullet3=6$
$(a^{30} + b^{30}) = (a^6 + b^6)\bullet(a^{24} - a^{18}b^6 + a^{12}b^{12} - a^6b^{18} + b^{24})$, a factor of 5 terms
Then to factor $(a^6 + b^6)$, $k=3$, $j=2$
$(a^6 + b^6) = (a^2 + b^2)\bullet(a^4 - a^2b^2 + b^4)$, a factor of 3 terms and the binomial factor $(a^2 + b^2)$

So putting all these factors together:
$a^{210} + b^{210} = (a^2 + b^2)\bullet(a^4 - a^2b^2 + b^4)\bullet(a^{24} - a^{18}b^6 + a^{12}b^{12} - a^6b^{18} + b^{24})$
$\bullet(a^{180} - a^{150}b^{30} + a^{120}b^{60} - a^{90}b^{90} + a^{60}b^{120} - a^{30}b^{150} + b^{180})$
This is four factors with 2, 3, 5, and 7 terms respectively.
(I will leave it to the reader to multiply all this out and verify it equals $a^{210} + b^{210}$. If you are really going to do this, start with the binomial factor and work up to the larger factors. If you cancel corresponding terms after each multiplication, you will have another binomial, and will have had a total of 3+5+7=15 corresponding pairs with opposite signs. Going the other way, combining largest factors first, you will end up with a polynomial of 210 terms with 104 matched pairs of terms with opposites signs scattered somewhere through out the 210-term polynomial.)

Consider a^{240}-b^{240}, n = $2^4\bullet3\bullet5$. Factoring this gives
[For 240, k=5,j=48] $a^{240}-b^{240} = (a^{48}-b^{48})\bullet(a^{192}+a^{144}b^{48}+a^{96}b^{96}+a^{48}b^{144}+b^{192})$
(For 48, k=3,j=16] $a^{48}-b^{48} = (a^{16}-b^{16})\bullet(a^{32}+a^{16}b^{16}+b^{32})$

$(a^{16}-b^{16}) = (a-b)\bullet(a+b)\bullet(a^2+b^2)\bullet(a^4+b^4)\bullet(a^8+b^8)$ (by formula II)

$(a^{32}+a^{16}b^{16}+b^{32}) = (a^{16}-a^8b^8+b^{16}) \bullet (a^8-a^4b^4+b^{16}) \bullet (a^4-a^2b^2+b^4) \bullet (a^2-ab+b^2)$
$\bullet(a^2+ab+b^2)$ (by formula III)

$(a^{192}+a^{144}b^{48}+a^{96}b^{96}+a^{48}b^{144}+b^{192}) = (a^{96}-a^{72}b^{24}+a^{48}b^{48}-a^{24}b^{72}+b^{96})$
$\bullet(a^{48}-a^{36}b^{12}+a^{24}b^{24}-a^{24}b^{36}+b^{48})$
$\bullet(a^{24}-a^{18}b^6+a^{12}b^{12}-a^6b^{18}+b^{24})$
$\bullet(a^{12}-a^9b^3+a^6b^6-a^3b^9+b^{12})\bullet(a^{12}+a^9b^3+a^6b^6+a^3b^9+b^{12})$ (by formula III)

$a^{240}-b^{240} = (a-b)\bullet(a+b)\bullet(a^2+b^2)\bullet(a^4+b^4)\bullet(a^8+b^8)$
$\bullet(a^{16}-a^8b^8+b^{16}) \bullet (a^8-a^4b^4+b^{16}) \bullet (a^4-a^2b^2+b^4) \bullet (a^2-ab+b^2) \bullet(a^2+ab+b^2)$
$\bullet$ $(a^{96}-a^{72}b^{24}+a^{48}b^{48}-a^{24}b^{72}+b^{96})\bullet(a^{48}-a^{36}b^{12}+a^{24}b^{24}-a^{24}b^{36}+b^{48})$
$\bullet(a^{24}-a^{18}b^6+a^{12}b^{12}-a^6b^{18}+b^{24})\bullet(a^{12}-a^9b^3+a^6b^6-a^3b^9+b^{12})$
$\bullet(a^{12}+a^9b^3+a^6b^6+a^3b^9+b^{12})$

This gives five factors each from $(a^{16}–b^{16})$, $(a^{32}+a^{16}b^{16}+b^{32})$, and $(a^{192}+a^{144}b^{48}+a^{96}b^{96}+a^{48}b^{144}+b^{192})$: four extras from the basic three: the binary and the two odd prime factors of n: 3 and 5.

The exponents of the factors based on the prime factors of n will vary from factor to factor. If there are two or more odd prime factors of n, they can be dealt with in any order. But since the exponents of the terms will be different when considering the prime factors in different orders, the different factors resulting from the different orders can all be used to find hidden factors. This will add an additional factor to $a^{240}–b^{240}$ for a total of sixteen factors and four additional factors to $a^{210} + b^{210}$ for a total of eight. (See the detailed factoring of $a^{105} + b^{105}$ below.)

To see the affect the order has on the result, consider the two factorings of $a^{15} + b^{15}$:
For j=3, k=5: $a^{15} + b^{15} = (a^3 + b^3) \bullet (a^{12} – a^9b^3 + a^6b^6 – a^3b^9 + b^{12})$
$= (a+b) \bullet (a^2 – ab + b^2) \bullet (a^{12} – a^9b^3 + a^6b^6 – a^3b^9 + b^{12})$
Product of last two factors:
$a^{14}–a^{11}b^3+a^8b^6–a^5b^9+a^2b^{12} – a^{13}b+a^{10}b^4–a^7b^7+a^4b^{10}–ab^{13} + a^{12}b^2–a^9b^5+a^6b^8–a^3b^{11}+b^{14}$
$= a^{14}–a^{13}b+a^{12}b^2–a^{11}b^3+a^{10}b^4–a^9b^5+a^8b^6–a^7b^7+a^6b^8–a^5b^9+a^4b^{10}–a^3b^{11}+a^2b^{12}–ab^{13}+b^{14}$

For j=5, k=3: $a^{15} + b^{15} = (a^5+b^5) \bullet (a^{10} – a^5b^5 + b^{10})$
$= (a+b) \bullet (a^4 – a^3b + a^2b^2 – ab^3 + b^4) \bullet (a^{10} – a^5b^5 + b^{10})$
Product of last two factors:
$a^{14}–a^9b^5+a^4b^{10} – a^{13}b+a^8b^6–a^3b^{11} + a^{12}b^2–a^7b^7+a^2b^{12} – a^{11}b^3+a^6b^8–ab^{13} + a^{10}b^4–a^5b^9+b^{14}$
$= a^{14}–a^{13}b+a^{12}b^2–a^{11}b^3+a^{10}b^4–a^9b^5+a^8b^6–a^7b^7+a^6b^8–a^5b^9+a^4b^{10}–a^3b^{11}+a^2b^{12}–ab^{13}+b^{14}$
This is the same as the previous, and with $(a+b)$ the two factoring are both factorings of $a^{15} + b^{15}$. The product of this with $(a+b)$ gives $a^{15} + b^{15}$. (The reader can verify this.)

Since both factorings are factorings of the same expression, but are different there must be some hidden factors. If one factoring gives $A \bullet B'$ and the other gives $B \bullet A'$ (A and B being the factors with lower order), then A must divide A' and B divide B'. $(a^2 – ab + b^2)$ must be a factor of $(a^{10} – a^5b^5 + b^{10})$. So another factor can be found by dividing $(a^{10} – a^5b^5 + b^{10})$ by $(a^2 – ab + b^2)$. Performing this division yields as quotient $(a^8 + a^7b – a^5b^3 – a^4b^4 – a^3b^5 + ab^7 + b^8)$. $(a^2 – ab + b^2)$ is not a factor of $(a^4 – a^3b + a^2b^2 – ab^3 + b^4)$ as has been verified by an attempted polynomial division. However, $(a^4 – a^3b + a^2b^2 – ab^3 + b^4)$ is a factor of $(a^{12} – a^9b^3 + a^6b^6 – a^3b^9 + b^{12})$ and yields $(a^8 + a^7b – a^5b^3 − a^4b^4 – a^3b^5 + ab^7 + b^8)$, the same quotient as $(a^{10} – a^5b^5 + b^{10})$ divided by $(a^2 – ab + b^2)$.

So $a^{15} + b^{15} = (a+b)$
$\bullet(a^2 – ab + b^2)$
$\bullet(a^4 – a^3b + a^2b^2 – ab^3 + b^4)$
$\bullet(a^8 + a^7b – a^5b^3 − a^4b^4 – a^3b^5 + ab^7 + b^8)$.

Is this last factor factorable also? Consider $2^{15} + 3^{15}$, when 2 is substituted for a and 3 for b, the four factors turn out to be 5•7•55•7174. But 7174 = 31•241. Both 31 and 241 are prime. Also, 55 = 5•11, so ultimately $2^{15} + 3^{15} = 5^2$•7•11•31•241 = 14,381,675. Even though 55 = 5•11 and the 5 is from $(a+b)$ and the 55 comes from $(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$, $(a+b)$ is not a factor of $(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$. So my conclusion is that this procedure guarantees finding a complete factoring of all binomials of the form $a^n \pm b^n$ except those of the form $a^c + b^c$, where $c=2^p$, for which there is no other factor. But it does not provide a method for finding all prime factors of such binomials in specific instances such as this case of $2^{15} + 3^{15}$.

Reconsidering the factoring of
$a^{60} + b^{60} = (a^4 + b^4) \bullet (a^8 - a^4b^4 + b^8) \bullet (a^{48} - a^{36}b^{12} + a^{24}b^{24} - a^{12}b^{36} + b^{48})$
presented earlier with k=5, $j=2^2 \bullet 3=12$,
let us now factor it with k=3 and $j=2^2 \bullet 5=20$:
$a^{60} + b^{60} = (a^{20} + b^{20}) \bullet (a^{40} - a^{20}b^{20} + b^{40})$
$= (a^4 + b^4) \bullet (a^{16} - a^{12}b^4 + a^8b^8 - a^4b^{12} + b^{16}) \bullet (a^{40} - a^{20}b^{20} + b^{40})$
$(a^{40} - a^{20}b^{20} + b^{40})$ divided by $(a^8 - a^4b^4 + b^8)$ gives the quotient $(a^{32} + a^{28}b^4 - a^{20}b^{12} - a^{16}b^{16} - a^{12}b^{20} + a^4b^{28} + b^{32})$. This is the same quotient as $(a^{48} - a^{36}b^{12} + a^{24}b^{24} - a^{12}b^{36} + b^{48})$ divided by $(a^{16} - a^{12}b^4 + a^8b^8 - a^4b^{12} + b^{16})$. So
$a^{60} + b^{60} = (a^4 + b^4)$
$\bullet (a^8 - a^4b^4 + b^8)$
$\bullet (a^{16} - a^{12}b^4 + a^8b^8 - a^4b^{12} + b^{16})$
$\bullet (a^{32} + a^{28}b^4 - a^{20}b^{12} - a^{16}b^{16} - a^{12}b^{20} + a^4b^{28} + b^{32})$.

Finding the extra factors of the $a^{60} - b^{60}$ from those of the form $\sum_{i=1}^{k} a^{2i} b^{2(k-i)}$ should be postponed until after the hidden factors are determined as the factored forms complicate the process. So doing this, the factors of $a^{60} - b^{60}$ are the same as for $a^{60} + b^{60}$ except that the factors other than $(a^4 - b^4)$ have all plus signs. So by the example of $a^{60} + b^{60}$:
$a^{60} - b^{60} = (a^{20} - b^{20}) \bullet (a^{40} + a^{20}b^{20} + b^{40})$ and $(a^{40} + a^{20}b^{20} + b^{40}) = (a^8 + a^4b^4 + b^8) \bullet (a^{32} - a^{28}b^4 + a^{20}b^{12} - a^{16}b^{16} + a^{12}b^{20} - a^4b^{28} + b^{32})$.
After that, the factors of the form $\sum_{i=1}^{k} a^{2i} b^{2(k-i)}$ can be factored giving:
$a^{60} - b^{60} = (a - b) \bullet (a + b) \bullet (a^2 + b^2)$
$\bullet (a^2 + ab + b^2) \bullet (a^2 - ab + b^2) \bullet (a^4 - a^2b^2 + b^4)$
$\bullet (a^4 + a^3b + a^2b^2 - ab^3 + b^4) \bullet (a^4 - a^3b + a^2b^2 - ab^3 + b^4) \bullet (a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8)$
$\bullet (a^{32} - a^{28}b^4 + a^{20}b^{12} - a^{16}b^{16} + a^{12}b^{20} - a^4b^{28} + b^{32})$. (No factors of hidden factor.)

If the two factorings are represented as $A \bullet B'$ and $B \bullet A'$, then $A' = A \bullet E$ and $B' = B \bullet E$, for some hidden factor E, the result is reduced by substitution to $A \bullet B \bullet E$.
So what happens when there are three odd prime factors of n? How many hidden factors are there? They can be combined pair-wise based on the reversal of the last two factors to determine a hidden factor. Each distinct pair of factorings can yield a hidden factor, which would make the number of hidden factors three when m = 3. As an example of this, consider $a^{105}+b^{105}$. $n = 3 \cdot 5 \cdot 7$, so there are six different orders of factoring: 3·5·7, 5·3·7, 3·7·5, 7·3·5, 5·7·3, 7·5·3. Each of these will produce a different set of factors, but pair-wise, by the above procedure, there should be a common factor between each pair yielding three additional factors. If the factors are $A \bullet B \bullet C$, there are six orders

to compute these factors: $A \bullet B' \bullet C''$, $B \bullet A' \bullet C''$, $A \bullet C' \bullet B''$, $C \bullet A' \bullet B''$, $B \bullet C' \bullet A''$, and $C \bullet B' \bullet A''$. The three different pairings are: $A \bullet B'$ with $B \bullet A'$, $A \bullet C'$ with $C \bullet A'$, and $B \bullet C'$ with $C \bullet B'$. In particular, for n=105, let A be the three-term polynomial ($T3$), B be the five-term polynomial ($T5$), and C be the seven-term polynomial ($T7$). Comparing AB' with $A'B$, we should get ABE_1 or $T3 \cdot T5 \cdot E_1$. Similarly, comparing AC' with CA', we should get ACE_2 or $T3 \cdot T7 \cdot E_2$, and comparing BC' with CB', we should get BCE_3 or $T5 \cdot T7 \cdot E_3$. It will be shown later that these factors all combine into seven factors of $a^{105}+b^{105}$: $(a+b)$, $T3$, $T5$, $T7$, E_1, E_2, and E_3. And the E's can be combined to give a final hidden factor for a total of eight.

Testing this hypothesis involves a lot of time-consuming polynomial division. But I have completed all the necessary polynomial divisions and verified the theory. The following is a summary of those calculations. [The reader may skip over the details of this example to the end for the algebraic analysis of this theory. However, seeing the details will clarify the process.]

So now we consider the complete Factoring of $a^{105} + b^{105}$:

$105 = 3 \cdot 5 \cdot 7$, $p=0$, $m=3$. We will factor $a^{105} + b^{105}$ several ways to look for all the hidden factors.

$k=7$, $j=15$:

$a^{105} + b^{105} = (a^{15} + b^{15}) \bullet (a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{75} + a^{30}b^{60} - a^{15}b^{75} + b^{90})$

$k=5$, $j=3$:

$a^{15} + b^{15} = (a + b) \bullet (a^2 - a^1b^1 + b^2) \bullet (a^{12} - a^9b^3 + a^6b^6 - a^3b^9 + b^{12})$

$k=3$, $j=5$:

$a^{15} + b^{15} = (a + b) \bullet (a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4) \bullet (a^{10} - a^5b^5 + b^{10})$

Dividing $(a^{10} - a^5b^5 + b^{10})$ by $(a^2 - a^1b^1 + b^2)$

gives $(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$.

Dividing $(a^{12} - a^9b^3 + a^6b^6 - a^3b^9 + b^{12})$ by $(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$ gives the same 8-degree polynomial.

So $a^{105} + b^{105} = (a + b)$

$\bullet (a^2 - a^1b^1 + b^2)$

$\bullet (a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$

$\bullet (a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$

$\bullet (a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{75} + a^{30}b^{60} - a^{15}b^{75} + b^{90})$

$k=5$, $j=21$:

$a^{105} + b^{105} = (a^{21} + b^{21}) \bullet (a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84})$

$k=7$, $j=3$:

$a^{21} + b^{21} = (a + b) \bullet (a^2 - a^1b^1 + b^2) \bullet (a^{18} - a^{15}b^3 + a^{12}b^6 - a^9b^9 + a^6b^{12} - a^3b^{15} + b^{18})$

$k=3$, $j=7$:

$a^{21} + b^{21} = (a + b) \bullet (a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6) \bullet (a^{14} - a^7b^7 + b^{14})$

Dividing $(a^{14} - a^7b^7 + b^{14})$ by $(a^2 - a^1b^1 + b^2)$

gives $(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$.

Dividing $(a^{18} - a^{15}b^3 + a^{12}b^6 - a^9b^9 + a^6b^{12} - a^3b^{15} + b^{18})$

by $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^{6)}$

gives the same 12-degree polynomial.

So $a^{105} + b^{105} = (a + b)$

• $(a^2 - a^1b^1 + b^2)$

• $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$

• $(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$

• $(a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84})$

k=3, j=35:

$a^{105} + b^{105} = (a^{35} + b^{35}) \bullet (a^{70} - a^{35}b^{35} + b^{70})$

k=7, j=5:

$a^{35} + b^{35} = (a + b) \bullet (a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4) \bullet (a^{30} - a^{25}b^5 + a^{20}b^{10} - a^{15}b^{15} + a^{10}b^{20} - a^5b^{25} + b^{30})$

k=5, j=7:

$a^{35} + b^{35} = (a + b) \bullet (a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6) \bullet (a^{28} - a^{21}b^7 + a^{14}b^{14} - a^7b^{21} + b^{28})$

Dividing $(a^{28} - a^{21}b^7 + a^{14}b^{14} - a^7b^{21} + b^{28})$ by $(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$ gives $(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$.

Dividing $(a^{30} - a^{25}b^5 + a^{20}b^{10} - a^{15}b^{15} + a^{10}b^{20} - a^5b^{25} + b^{30})$ by $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$ also gives the same 24-degree polynomial

So $a^{105} + b^{105} = (a + b)$

• $(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$

• $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$

• $(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$

•$(a^{70} - a^{35}b^{35} + b^{70})$

If we can put all the hidden factors in together, we would get the following, but can we justify combining them together into one set of factors?

$a^{105} + b^{105} = (a+b)$

• $(a^2 - a^1b^1 + b^2)$

• $(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$

• $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$

• $(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$

• $(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$

• $(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$.

The degree of the product would be 1+2+4+6+8+12+24 = 57. This is far from the required 105, so something of degree 48 must be missing. There must be something more contained in the polynomials $(a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{75} + a^{30}b^{60} - a^{15}b^{75} + b^{90})$, $(a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84})$, and $(a^{70} - a^{35}b^{35} + b^{70})$

The factor sets of $a^{105} + b^{105}$ are equal factorings of $a^{105} + b^{105}$ and are therefore equal to each other. (In specifying these sets I have eliminated the factor $(a+b)$ from each set since it is common to all of them.)

Set 1: $(a^2 - a^1b^1 + b^2)$
$\bullet(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$
$\bullet(a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{45} + a^{30}b^{60} - a^{15}b^{75} + b^{90})$
=
Set 2: $(a^2 - a^1b^1 + b^2)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$\bullet(a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84})$
=
Set 3: $(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} +$
$a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$
$\bullet(a^{70} - a^{35}b^{35} + b^{70})$

In these sets, the basic factors for the 3 odd prime factors of 105 (3,5,7) are:
$(a^2 - a^1b^1 + b^2)$
$(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$

And the three hidden factors are:
$(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$
$(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} +$
$a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$

The leftover factors that will be used to combine everything are:
$(a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{45} + a^{30}b^{60} - a^{15}b^{75} + b^{90})$
$(a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84})$
$(a^{70} - a^{35}b^{35} + b^{70})$

So comparing each pair of sets by eliminating a common factor, we have from sets 1 & 2:
$(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$
$\bullet(a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{45} + a^{30}b^{60} - a^{15}b^{75} + b^{90})$
= $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$\bullet(a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84})$

and from sets 2 & 3:
$(a^2 - a^1b^1 + b^2)$
$\bullet(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$\bullet(a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84})$
= $(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} +$
$a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$
$\bullet(a^{70} - a^{35}b^{35} + b^{70})$

and from sets 1 & 3:
$(a^2 - a^1b^1 + b^2)$

$\bullet(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$
$\bullet(a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{45} + a^{30}b^{60} - a^{15}b^{75} + b^{90})$
= $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$
$\bullet(a^{70} - a^{35}b^{35} + b^{70})$

Each of these pairs of sets consists of three different factors, but the products in each set are equal. These different equal factorings imply that there are other common factors. Large polynomial divisions yield $(a^{90} - a^{75}b^{15} + a^{60}b^{30} - a^{45}b^{45} + a^{30}b^{60} - a^{15}b^{75} + b^{90}) = (a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6) \bullet$ [a factor of 25 terms from a^{84} to b^{84}]
and $(a^{84} - a^{63}b^{21} + a^{42}b^{42} - a^{21}b^{63} + b^{84}) = (a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4) \bullet$ [a factor of 33 terms from a^{80} to b^{80}]
and $(a^{70} - a^{35}b^{35} + b^{70}) = (a^2 - a^1b^1 + b^2) \bullet$ [a factor of 47 terms from a^{68} to b^{68}]
Making these substitutions in sets 1, 2, and 3 gives:

Set 1: $(a^2 - a^1b^1 + b^2)$
$\bullet(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$
$\bullet$ [a factor of 25 terms from a^{84} to b^{84}]
=
Set 2: $(a^2 - a^1b^1 + b^2)$
$\bullet(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$\bullet$ [a factor of 33 terms from a^{80} to b^{80}]
=
Set 3: $(a^2 - a^1b^1 + b^2)$
$\bullet(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$
$\bullet$ [a factor of 47 terms from a^{68} to b^{68}]

So there must be some relation between the larger factors and the factors from analyzing the different factorings pair-wise. The 25-term factor of degree 84 must contain the 9-term degree 12 factor and the 17-term degree 24 factor. Similarly, the 33-term degree 80 factor must contain the 7-term degree 8 factor and the 17-term degree 24 factor. And the 47-term degree 68 factor must contain the 7-term degree 8 factor and the 9-term degree 12 factor. It will be sufficient to verify one of these so I will show that the 25-term degree 84 factor has the 9-term degree 12 factor and the 17-term degree 24 factor as factors. Note that 84-12-24 = 48, 80-8-24 = 48, and 68-8-12 = 48, the missing degree that was noted above. So the combination of these sets must lead to the missing 48-degree polynomial factor.

Let's perform some more polynomial division and test this theory.

The large factor of the first set is $(a^{84} + a^{83}b^1 - a^{77}b^7 - a^{76}b^8 + a^{70}b^{14} - a^{68}b^{16} - a^{63}b^{21} + a^{61}b^{23} + a^{56}b^{28} + a^{53}b^{31} - a^{49}b^{35} - a^{46}b^{38} + a^{42}b^{42} - a^{38}b^{46} - a^{35}b^{49} + a^{31}b^{53} + a^{28}b^{56} + a^{23}b^{61} - a^{21}b^{63} - a^{16}b^{68} + a^{14}b^{70} - a^8b^{76} - a^7b^{77} + a^1b^{83} + b^{84})$

The large factor of the second set is $(a^{80} + a^{79}b^1 - a^{75}b^5 - a^{74}b^6 + a^{70}b^{10} + a^{69}b^{11} - a^{65}b^{15} - a^{64}b^{16} + a^{60}b^{20} - a^{58}b^{22} - a^{55}b^{25} + a^{53}b^{27} + a^{50}b^{30} - a^{48}b^{32} - a^{45}b^{35} + a^{43}b^{37} + a^{40}b^{40} + a^{37}b^{43} - a^{35}b^{45} - a^{32}b^{48} + a^{30}b^{50} + a^{27}b^{53} - a^{25}b^{55} - a^{22}b^{58} + a^{20}b^{60} - a^{16}b^{64} - a^{15}b^{65} + a^{11}b^{69} + a^{10}b^{70} - a^6b^{74} - a^5b^{75} + a^1b^{79} + b^{80})$

The large factor of the third set is $(a^{68} + a^{67}b^1 - a^{65}b^3 - a^{64}b^4 + a^{62}b^6 + a^{61}b^7 - a^{59}b^9 - a^{58}b^{10} + a^{56}b^{12} + a^{55}b^{13} - a^{53}b^{15} - a^{52}b^{16} + a^{50}b^{18} + a^{49}b^{19} - a^{47}b^{21} - a^{46}b^{22} + a^{44}b^{24} + a^{43}b^{25} - a^{41}b^{27} - a^{40}b^{28} + a^{38}b^{30} + a^{37}b^{31} - a^{35}b^{33} - a^{34}b^{34} - a^{33}b^{35} + a^{31}b^{37} + a^{30}b^{38} - a^{28}b^{40} - a^{27}b^{41} + a^{25}b^{43} + a^{24}b^{44} - a^{22}b^{46} - a^{21}b^{47} + a^{19}b^{49} + a^{18}b^{50} - a^{16}b^{52} - a^{15}b^{53} + a^{13}b^{55} + a^{12}b^{56} - a^{10}b^{58} - a^9b^{59} + a^7b^{61} + a^6b^{62} - a^4b^{64} - a^3b^{65} + a^1b^{67} + b^{68})$

Let's test this theory with set one. The large 84-degree factor has as a factor the 24-degree factor from set three:

$(a^{84} + a^{83}b^1 - a^{77}b^7 - a^{76}b^8 + a^{70}b^{14} - a^{68}b^{16} - a^{63}b^{21} + a^{61}b^{23} + a^{56}b^{28} + a^{53}b^{31} - a^{49}b^{35} - a^{46}b^{38} + a^{42}b^{42} - a^{38}b^{46} - a^{35}b^{49} + a^{31}b^{53} + a^{28}b^{56} + a^{23}b^{61} - a^{21}b^{63} - a^{16}b^{68} + a^{14}b^{70} - a^8b^{76} - a^7b^{77} + a^1b^{83} + b^{84}) =$
$(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$
$\bullet\ (a^{60} + a^{55}b^5 - a^{45}b^{15} - a^{40}b^{20} + a^{30}b^{30} - a^{20}b^{40} - a^{15}b^{45} + a^5b^{55} + b^{60})$

This 60-degree factor then has as a factor the 12-degree factor from set two:

$(a^{60} + a^{55}b^5 - a^{45}b^{15} - a^{40}b^{20} + a^{30}b^{30} - a^{20}b^{40} - a^{15}b^{45} + a^5b^{55} + b^{60}) =$
$(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$\bullet\ (a^{48} - a^{47}b^1 + a^{46}b^2 + a^{43}b^5 - a^{42}b^6 + 2a^{41}b^7 - a^{40}b^8 + a^{39}b^9 + a^{36}b^{12} - a^{35}b^{13} + a^{34}b^{14} - a^{33}b^{15} + a^{32}b^{16} - a^{31}b^{17} - a^{28}b^{20} - a^{26}b^{22} - a^{24}b^{24} - a^{22}b^{26} - a^{20}b^{28} - a^{17}b^{31} + a^{16}b^{32} - a^{15}b^{33} + a^{14}b^{34} - a^{13}b^{35} + a^{12}b^{36} + a^9b^{39} - a^8b^{40} + 2a^7b^{41} - a^6b^{42} + a^5b^{43} + a^2b^{46} - a^1b^{47} + b^{48})$

This means that we have now merged all of the three sets into one with all of the previous hidden factors included and have found the missing 48-degree polynomial factor:

$a^{105} + b^{105} = (a+b) \bullet (a^2 - a^1b^1 + b^2)$
$\bullet(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$
$\bullet(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$\bullet(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$

$\bullet(a^{48} - a^{47}b^1 + a^{46}b^2 + a^{43}b^5 - a^{42}b^6 + 2a^{41}b^7 - a^{40}b^8 + a^{39}b^9 + a^{36}b^{12} - a^{35}b^{13} + a^{34}b^{14} - a^{33}b^{15} + a^{32}b^{16} - a^{31}b^{17} - a^{28}b^{20} - a^{26}b^{22} - a^{24}b^{24} - a^{22}b^{26} - a^{20}b^{28} - a^{17}b^{31} + a^{16}b^{32} - a^{15}b^{33} + a^{14}b^{34} - a^{13}b^{35} + a^{12}b^{36} + a^9b^{39} - a^8b^{40} + 2a^7b^{41} - a^6b^{42} + a^5b^{43} + a^2b^{46} - a^1b^{47} + b^{48})$

[All of the preceding is the result of many days of work resulting in seven pages of dense hand written polynomial divisions carefully checked at each stage for errors.]

I investigated whether the remaining 48-degree factor could actually be a combination of multiple occurrences of the original factors, possibly four of each of them, two being generated each with the second and third ordering sets:

$(a^2 - a^1b^1 + b^2)$
$\bullet(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$

The product of four of each of these three would account for the 48-degree polynomial. Possibly these additional factors were created each time we factored the original polynomial using a different ordering of the prime factors of 105.

To test this, I attempted to divide the 33-term 48-degree polynomial by $(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$, but that failed. Also I tried dividing it by $(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$, and that, too, failed. Neither of these are factors of that polynomial. I noticed some symmetry in that polynomial, and thought it might be factored by grouping into a three-term polynomial and an eleven-term polynomial, (33=3•11). But while the first and third eleven-term parts are identical, the middle one is very different, so factoring by grouping does not work either.

At this point I have no further explanation of the origin of the 33-term 48-degree factor and consider it a deeper level hidden factor, revealed when all the other factors are integrated into one set of factors. So from this analysis the final factors of $a^{105} + b^{105}$ are:

$(a+b)$
$\bullet(a^2 - a^1b^1 + b^2)$
$\bullet(a^4 - a^3b^1 + a^2b^2 - a^1b^3 + b^4)$
$\bullet(a^6 - a^5b^1 + a^4b^2 - a^3b^3 + a^2b^4 - a^1b^5 + b^6)$
$\bullet(a^8 + a^7b^1 - a^5b^3 - a^4b^4 - a^3b^5 + a^1b^7 + b^8)$
$\bullet(a^{12} + a^{11}b^1 - a^9b^3 - a^8b^4 + a^6b^6 - a^4b^8 - a^3b^9 + a^1b^{11} + b^{12})$
$\bullet(a^{24} + a^{23}b^1 - a^{19}b^5 - a^{18}b^6 - a^{17}b^7 - a^{16}b^8 + a^{14}b^{10} + a^{13}b^{11} + a^{12}b^{12} + a^{11}b^{13} + a^{10}b^{14} - a^8b^{16} - a^7b^{17} - a^6b^{18} - a^5b^{19} + a^1b^{23} + b^{24})$
$\bullet(a^{48} - a^{47}b^1 + a^{46}b^2 + a^{43}b^5 - a^{42}b^6 + 2a^{41}b^7 - a^{40}b^8 + a^{39}b^9 + a^{36}b^{12} - a^{35}b^{13} + a^{34}b^{14} - a^{33}b^{15} + a^{32}b^{16} - a^{31}b^{17} - a^{28}b^{20} - a^{26}b^{22} - a^{24}b^{24} - a^{22}b^{26} - a^{20}b^{28} - a^{17}b^{31} + a^{16}b^{32} - a^{15}b^{33} + a^{14}b^{34} - a^{13}b^{35} + a^{12}b^{36} + a^9b^{39} - a^8b^{40} + 2a^7b^{41} - a^6b^{42} + a^5b^{43} + a^2b^{46} - a^1b^{47} + b^{48})$

The following is an algebraic proof of the preceding result for $a^{105} + b^{105}$. Let A, B, C be the basic prime factors of $a^{105}+b^{105} = Q$, the polynomial involved, based on the odd prime factors of n where $n = 105$, $m = 3$, in this case. In any factoring, let A', B', C' be the second factors, and let A'', B'', C'' be the third large degree factors initially obtained. In this discussion all binomial factors of the form of $a^c - b^c$ and $a^c + b^c$, where $c=2^p$ are omitted.

As noted previously, $Q = AB'C'' = BA'C''$ by the two factorings resulting from reversing the factorings from the last two primes. Since $AB' = BA'$, there must be a common factor between these. Since the degree of A < degree of A' and the degree of B < degree of B', A must be a factor of A' and B a factor of B'. $A' = AE_1$ and $B' = BE_1$. By substitution $Q = ABE_1C''$ and $Q = B'A'E_1C''$ for some additional factor E_1. Choose $Q = ABE_1C''$.
Also $Q = AC'B'' = CA'B''$, so similarly, $AC' = CA'$. $C' = CE_2$ and $A' = AE_2$, and $Q = ACE_2B''$.
In the same way $Q = BC'A'' = CB'A''$, $C' = CE_3$, $B' = BE_3$, and $Q = BCE_3A''$.

Since all of these are equal, $Q = ABE_1C'' = ACE_2B'' = BCE_3A''$, C'' must combine C, E_2, and E_3, B'' must combine B, E_1, and E_3, A'' must combine A, E_1, and E_2. So $Q = ABE_1(CE_2E_3G) = ACE_2(BE_1E_3G) = BCE_3(AE_1E_2G) = ABCE_1E_2E_3G$ for some new hidden factor G. Since G is the only difference between each of these, it has to be the same quantity in each case.

In this case, the 33-term 48-degree factor is the lower level hidden factor G revealed when the other first level hidden factors are combined. What we have shown is that with one odd prime factor of n, we get one factor in addition to the binary factor, e.g. $a^3 + b^3 = (a + b) \bullet (a^2 - a^1b^1 + b^2)$. With two odd prime factors of n we get two factors in addition to the binary factor and one hidden factor found by comparing different orders of factoring. With three odd prime factors of n, we get three factors, three hidden factors from comparing the three factors pair-wise, and one deeper hidden factor revealed when we combine the three hidden factors into one set. This has been demonstrated above at this point in this paper.

The binomial coefficients describe the process of revealing the number of factors. When n has two prime factors, $\binom{2}{1}$ =2 is the number of original factors $\binom{2}{2}$ =1 hidden factor found by comparing the two original factors. When there are three odd prime factors of n, there are $\binom{3}{1}$ =3 factors, $\binom{3}{2}$ =3 hidden factors found by comparing the original factors two at a time, and $\binom{3}{3}$ =1 additional hidden factor comparing all three original factors and the hidden factors found by comparing them pair-wise. So with four different odd prime factors of n, we would get four factors ($\binom{4}{1}$ =4), six hidden factors comparing these pair-wise ($\binom{4}{2}$=6), four deeply hidden factors ($\binom{4}{3}$ =4) when we merge the six hidden factors together and one further hidden factor ($\binom{4}{4}$=1) that emerges from the combination of the four deeply hidden factors into one set. For example there are six ways of comparing four factors two at a time: *AB*, *A*C, *AD*, *B*C, *BD*, and C*D*. Each of these leads to a hidden factor. Hence $\binom{4}{2}$ = 6 describes the number of first level hidden factors in the case *m*=4. When we take these and add another factor we get $\binom{4}{3}$ = 4 hidden factors at the next level.

Summarizing this for different numbers of discrete odd prime factors of *n*:

m=0: 1 set of binary factors (1)
m=1: 1 set of binary factors, 1 factor (2)
m=2: 1 set of binary factors, 2 factors, 1 hidden factor (4)
m=3: 1 set of binary factors, 3 factors, 3 hidden factors, 1 deeply hidden factor (8)
m=4: 1 set of binarys, 4 factors, 6 hiddens, 4 deep hiddens, 1 very hidden (16)

Perhaps analyzing $a^{1155} + b^{1155}$ (1155 = 3•5•7•11, *n* = the product of four odd prime factors), would verify the above. But this represents an exponential increase in the amount and size of polynomial divisions that must be performed and so will not be attempted, but we will analyze it algebraically below.

This pattern follows that of Pascal's Triangle for the coefficients of $(a+b)^n$. There is similarity between the ways of comparing the original factors pair-wise and the binomial expansion of the power of two terms. This implies the number of factors of $a^n \pm b^n$ when $n = 2^p\prod_{i=1}^{m} f_i$ is $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \ldots + \binom{m}{m} = 2^m$, not counting the separate binary factors resulting from $a^n - b^n$, but only counting all the binary factors as a group corresponding to $\binom{m}{0}$. But remembering that duplicate factors do not generate hidden factors (since the order of identical factors makes no difference and hence no hidden factor), the formula must be adjusted accordingly.

For $n = 2^p\prod_{i=1}^{m} f_i$, and *m*=*r*+*s*, where the *r* factors are distinct and the *s* factors are duplicates of some of the *r* factors, the number of factors of $a^n + b^n$ is $2^r + s$ and the number of factors of $a^n - b^n$ is $2^r + s + p$.

This analysis does not include the factors that arise from factors of the form $\sum_{i=0}^{k} a^{2i}b^{2(k-i)}$.

These factors only arise when there is a factor of the form $a^n - b^n$ where *n* contains at least one odd prime. This procedure will generate *p* new factors for each of the odd prime factors of *n*.

So the formula for the number of factors of $a^n - b^n$ is $2^r + s + p \bullet (m+1)$.
The number of factors of $a^n + b^n$ is $2^r + s$.

We can now verify the number of hidden factors in the case of $a^n + b^n$ with four distinct odd prime factors of *n* by representing each factor as a single variable and performing the same algebra that was used in the case of three odd prime factors. Let the four distinct odd prime factors lead to the polynomial factors A, B, C, and D of Q, the degree n polynomial. There are 4! = 24 ways of ordering these factors. Representing the order of the factors with accents (since the exponents will be different depending on when the factor is determined) with the largest set of accents for the first factor determined and the least number of accents for the last, we have $AB'C''D'''$, $BA'C''D'''$, etc. Holding the C'' and D'' constant results in twelve combinations involving the pairs A-B, A-C, A-D, B-C, B-D, C-D. For each of these there are two orders, e.g. AB' and BA'. By the procedure described above, each of these pairs produces a hidden factor. I will refer to them as E_i: $ABE_1C''D'''$, $ACE_2C''D'''$, $ADE_3C''D'''$, $BCE_4C''D'''$, $BDE_5C''D'''$, $CDE_6C''D'''$. Each of these occurs twice as the C'' and D''' can be reversed, e.g. $ABE_1C''D'''$ and $ABE_1D''C'''$.

First consider $ABE_1C''D'''$, $ACE_2B''D'''$, and $BCE_4A''D'''$. Since all of these are equal, $C'' = CE_2E_4$, $B'' = BE_1E_4$, and $A'' = AE_1E_2$, so they can be combined leading to a new hidden factor F_1. So $Q = ABCE_1E_2E_4F_1D'''$.
Similarly from the group three factorings based on factors ending with C''', $Q = ABDE_1E_3E_5F_2C'''$. From the B''' group we get $Q = ACDE_2E_3E_6F_3B'''$, and from the A'' group we get $Q = BCDE_4E_5E_6F_4A'''$.

Since $Q = ABCE_1E_2E_4F_1D''' = ABDE_1E_3E_5F_2C'''= ACDE_2E_3E_6F_3\,B''' = BCDE_4E_5E_6F_4\,A''$, each of the A''', B''', C''', and D''' must contain what that factoring lacks from the other three plus a new hidden factor G. So $D'''=DE_3E_5E_6F_2F_3F_4G$, $C'''=CE_2E_4E_6F_1F_3F_4G$, $B'''=BE_1E_4E_5F_1F_2F_4G$, and $A'''=AE_1E_2E_3F_1F_2F_3G$.
Now combining all of these together gives $Q = ABCDE_1E_2E_3E_4\,E_5\,E_6F_1F_2F_3F_4G$, 16 factors including the binary factor group, as per the formula proposed above including the binomial coefficients for the number of factors at each level: 1 binaries, 4 originals, 6 E's, 4 F's , and1 G.

One final example that illustrates everything I have presented in this paper: The factoring of $a^{252} - b^{252}$. In this case, $n = 252 = 2^2 \bullet 3^2 \bullet 7$, m=3, p=2, r=2, s=1

$a^{252} - b^{252} = (a^{36} - b^{36}) \bullet (a^{216} + a^{180}b^{36} + a^{144}b^{72} + a^{108}b^{108} + a^{72}b^{144} + a^{36}b^{180} + b^{216})$ (by formula I, k=7, j=36.)

$a^{36} - b^{36} = (a^4 - b^4) \bullet (a^{32} + a^{28}b^4 + a^{24}b^8 + a^{20}b^{12} + a^{16}b^{16} + a^{12}b^{20} + a^8b^{24} + a^4b^{28} + b^{32})$ (by formula I, k=9, j=4.)

$a^{36} - b^{36} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^{32} + a^{28}b^4 + a^{24}b^8 + a^{20}b^{12} + a^{16}b^{16} + a^{12}b^{20} + a^8b^{24} + a^4b^{28} + b^{32})$ (by formula II.)

At this point we could apply formulas III and IV, but since this would complicate the polynomial division to find the hidden factor, these factorings will be postponed until that factor is determined.

$a^{252} - b^{252} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^{32} + a^{28}b^4 + a^{24}b^8 + a^{20}b^{12} + a^{16}b^{16} + a^{12}b^{20} + a^8b^{24} + a^4b^{28} + b^{32}) \bullet (a^{216} + a^{180}b^{36} + a^{144}b^{72} + a^{108}b^{108} + a^{72}b^{144} + a^{36}b^{180} + b^{216})$

Also

$a^{252} - b^{252} = (a^{28} - b^{28}) \bullet (a^{224} + a^{196}b^{28} + a^{168}b^{56} + a^{140}b^{84} + a^{112}b^{112} + a^{84}b^{140} + a^{56}b^{168} + a^{28}b^{196} + b^{224})$ (by formula I, k=9, j=28.)

$a^{28} - b^{28} = (a^4 - b^4) \bullet (a^{24} + a^{20}b^4 + a^{16}b^8 + a^{12}b^{12} + a^8b^{16} + a^4b^{20} + b^{24})$ (by formula I, k=7, j=4.)

$a^{252} - b^{252} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^{24} + a^{20}b^4 + a^{16}b^8 + a^{12}b^{12} + a^8b^{16} + a^4b^{20} + b^{24}) \bullet (a^{224} + a^{196}b^{28} + a^{168}b^{56} + a^{140}b^{84} + a^{112}b^{112} + a^{84}b^{140} + a^{56}b^{168} + a^{28}b^{196} + b^{224})$ (by formula II.)

By dividing the degree 24 factor into the degree 216 factor or dividing the degree 32 factor into the degree 224 factor, we get a 31-term degree 192 hidden factor. Note: the degree 192 factor can also be found by ignoring the 2^2 factors of n and dividing the resulting degree 54 factor (216/4=54) by the degree 6 factor (24/4=6) and then multiplying each of the exponents of the resulting degree 48 factor by 4 to get the required degree 192 factor, which is what I did.

So,

$a^{252} - b^{252} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^{24} + a^{20}b^4 + a^{16}b^8 + a^{12}b^{12} + a^8b^{16} + a^4b^{20} + b^{24}) \bullet (a^{32} + a^{28}b^4 + a^{24}b^8 + a^{20}b^{12} + a^{16}b^{16} + a^{12}b^{20} + a^8b^{24} + a^4b^{28} + b^{32}) \bullet$ (a 31-term degree 192 factor)

Now we can make the formula III and IV factorings.

$a^{252} - b^{252} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^{24} + a^{20}b^4 + a^{16}b^8 + a^{12}b^{12} + a^8b^{16} + a^4b^{20} + b^{24}) \bullet (a^8 + a^4b^4 + b^8) \bullet (a^{24} + a^{12}b^{12} + b^{24}) \bullet$ (a 31-term degree 192 factor) (by formula IV)

$a^{252} - b^{252} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6) \bullet (a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6) \bullet (a^{12} - a^{10}b^2 + a^8b^4 - a^6b^6 + a^4b^8 - a^2b^{10} + b^{12}) \bullet (a^8 + a^4b^4 + b^8) \bullet (a^{24} + a^{12}b^{12} + b^{24}) \bullet$ (a 31-term degree 192 factor) (by formula III twice)

$a^{252} - b^{252} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6) \bullet (a^6 - a^5b + a^{14}b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6) \bullet (a^{12} - a^{10}b^2 + a^8b^4 - a^6b^6 + a^4b^8 - a^2b^{10} + b^{12}) \bullet (a^2 + ab + b^2) \bullet (a^2 - ab + b^2) \bullet (a^4 - a^2b^2 + b^4) \bullet (a^{24} + a^{12}b^{12} + b^{24}) \bullet$ (a 31-term degree 192 factor) (by formula III twice.)

$a^{252} - b^{252} = (a - b) \bullet (a + b) \bullet (a^2 + b^2) \bullet (a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6) \bullet (a^6 - a^5b + a^{14}b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6) \bullet (a^{12} - a^{10}b^2 + a^8b^4 - a^6b^6 + a^4b^8 - a^2b^{10} + b^{12}) \bullet (a^2 + ab + b^2) \bullet (a^2 - ab + b^2) \bullet (a^4 - a^2b^2 + b^4) \bullet (a^6 + a^3b^3 + b^6) \bullet (a^6 - a^3b^3 + b^6) \bullet (a^{12} - a^6b^6 + b^{12}) \bullet$ (a 31-term degree 192 factor) (by formula III twice.)

Note adding the degrees of each factor: 1+1+2+6+6+12+2+2+4+6+6+12+192 = 252.

Also note that for the number of factors formula, $2^r + s + p \bullet (m+1)$, r=2, s=1, p=2, m=3, so the number of factors is $2^2 + 1 + 2\bullet(3+1) = 4 + 1 + 8 = 13$. This is the number of factors shown above for $a^{252} - b^{252}$.

In case you are interested, here is the 31-term degree 192 factor:

$a^{192} - a^{188}b^4 + a^{164}b^{28} - a^{160}b^{32} + a^{156}b^{36} - a^{152}b^{40} + a^{136}b^{56} - a^{132}b^{60} + a^{128}b^{64} - a^{124}b^{68} + a^{120}b^{72} - a^{116}b^{76} + a^{108}b^{84} - a^{104}b^{88} + a^{100}b^{92} - a^{96}b^{96} + a^{92}b^{100} - a^{88}b^{104} + a^{84}b^{108} - a^{76}b^{116} + a^{72}b^{120} - a^{68}b^{124} + a^{64}b^{128} - a^{60}b^{132} + a^{56}b^{136} - a^{40}b^{152} + a^{36}b^{156} - a^{32}b^{160} + a^{28}b^{164} - a^4b^{188} + b^{192}$

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