An Algorithm to Reverse the Generalized Factorials Process

Master's Report

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April 5, 2019

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Abstract

We had previously analyzed Dr. Manjul Bhargava's paper, *The Factorial Function and Generalizations*. Dr. Bhargava generalizes the factorial function for any subset of the integers. The goal of this research paper was to utilize the methods employed in Dr. Bhargava's paper to find combinatorial patterns within the factorial function. Mathematica code was generated to compute generalized factorials given a subset of the integers based on p-orderings and p-sequences. By the end of the research, algorithms were proposed to reverse this process, that is, given a sequence of generalized factorials, generating the subset it came from originally.

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Chapter 1

Introduction

Dr. Manjul Bhargava wrote a paper on the "Factorial Function and Generalizations" in 2000. The factorial function, defined on the integers, is the product of consecutive integers less than or equal to itself. More formally, it is defined as follows:

$$n! = \prod_{k=1}^{n} k = n(n-1)(n-2)...(3)(2)(1)$$

The factorial function occurs frequently in mathematics, especially in combinatorics and number theory. The motivating question of Dr. Bhargava's paper was as follows: "Is there some other function–some generalized function–that we could change the ordinary factorials,so [certain number-theoretic theorems] would still remain true?" [1]. The author then proceeded to provide a methodology to generate factorials on any subset of the integers. At the end of the paper, the author posed an intriguing question, "What is the 'binomial theorem' for generalized binomial coefficients?" [1].

Two years ago, I had conducted research on the "Generalization of the Binomial Theorem and Square Number Patterns." We found patterns in square numbers and utilized the Binomial Theorem to generalize these patterns. While extending the formulas from integers to real numbers, combinatorial identities were utilized. The initial **purpose** of this research project was to compare the combinatorial identities that occured in our prior square number research and relate it to the properties of the factorial function.

The research process occured in three phases. In the first phase, I closely analyzed Dr. Bhargava's paper, familiarized myself with different topics in combinatorics, and conducted literary research on papers relating to the Generalized Factorials and binomial coefficients.

In phase two, Mathematica code was generated based on Bhargava's methodology of generating factorials on any subset of the integers. First, we checked our solutions from known examples, but later, we experimented with different subsets. Throughout this experimentation process, several patterns were found and analyzed. This code became a tool for us through the end of the research.

In phase three, we did original research and attempted to implement a new algorithm. Earier, we had looked at another work [10] in which particular generalized factorials were connected to a power series. If an algorithm could be made in which the factorials and power series were directly connected, then we could apply several calculus methods on the generalized factorials, which would increase the scope of its applications. In order to achieve this, we had to attempt to solve several sub-problems. One of these problems, reversing the Bhargava factorial process, became the focus of this research project.

1.1 Phase I: Background and Literary Research

The major inspiration of the paper comes from Dr. Manjul Bhargava's *Factorial Function and Generalization.* In the following section, we will emphasize important features from that paper. We want to find what it means to be a factorial for any subset $S \subset \mathbb{Z}$. The procedure is described below:

1. For a subset S of the integers, fix a prime number, p.
2. A **p-ordering** of S is a sequence $\{a_i\}_{i=0}^{\infty}$ of elements of S that is formed by the following methodology:

a. Choose an element $a_0 \in S$.

b. Choose $a_1 \in S$ that minimizes the highest power of p that divides $a_1 - a_0$.

c. Choose $a_2 \in S$ that minimizes the highest power of p that divides $(a_2-a_0)(a_2-a_1)$.

d. Similarly, choose $a_k \in S$ that minimizes the highest power of p that divides $(a_k - a_0)(a_k - a_1)\cdots(a_k - a_{k-1})$.

3. Repeat this process for all the prime numbers p.
4. The set $\{a_0, a_1, a_2, ...\}$ is referred to as the 'p-ordering' of the set S. Note, p-orderings need not be unique.
5. Let $v_k(S,p)$ be the highest power of p that divides $(a_k - a_0)(a_k - a_1)\cdots(a_k - a_{k-1})$. Then, $\{v_0(S,p), v_1(S,p), ...\}$ is referred to as the associated 'p-sequence.' (Assume, $v_0(S,p) = 1$). Note, regardless of the choice of the p-ordering, the p-sequence is unique.
6. The generalized factorial for the integer k (associated with the infinite set S) is defined as follows:
$$k!_S = \prod_p v_k(S,p) \tag{1.1}$$
where the product is taken over all prime numbers p[1].

1.2 Example

1. Consider the set of prime numbers. Let $S = \{2, 3, 5, 7, ...\} \subset \mathbb{Z}$.

   Fix prime, $p = 2$
2. The first few iterations of the p-orderings and p-sequences are given below.

a. Choose $a_0 = 19$ from S.

b. Choose a_1 that minimizes that highest power of 2 that divides $a_1 - 19$. The only possibility is $a_1 = 2$.

$2^m | 19 - 2 = -17 \Rightarrow 2^0 = 1 | -17$. Hence, $v_1(P, 2) = 1$

c. Choose a_2 that minimizes the highest power of 2 that divides $(a_2 - 19)(a_2 - 2)$ After some experimentation, it can be seen that $a_2 = 5$ and $v_2(P, 2) = 2$.

d. Choose a_3 that minimizes the highest power of 2 that divides $(a_3 - 19)(a_3 - 2)(a_3 - 5)$. It can be seen that $a_3 = 17$ and $v_3(P, 3) = 8$

e. Continue this process for the elements in S and observe the related p-orderings and p-sequences.

3. The p-ordering and p-sequence for $p = 2$ is as follows:

   P-ordering: $\{19, 2, 5, 17, 23, 31, ...\}$

   P-sequences: $\{1, 1, 2, 8, 16, 128, ...\}$

4. Repeat this process for all prime numbers p.

   A p-ordering and p-sequence for $p = 3$ is as follows:

   P-ordering: $\{2, 3, 7, 5, 13, 17, 19, ...\}$

   P-sequence: $\{1, 1, 1, 3, 3, 9, ...\}$

5. A p-ordering and p-sequence for $p = 5$ is as follows:

   P-ordering: $\{2, 3, 5, 19, 11, 7, 13, ...\}$

   P-sequence: $\{1, 1, 1, 1, 1, 5, ...\}$

6. Compute the generalized factorial. Denote the primes by P.

$$k!_P = \prod_p v_k(P, p)$$

	$p=2$	$p=3$	$p=5$	$p=7$	...	$k!_p$
$k=0$	1	1	1	1	...	1
$k=1$	1	1	1	1	...	1
$k=2$	2	1	1	1	...	2
$k=3$	8	3	1	1	...	24
$k=4$	16	3	1	1	...	48
$k=5$	128	9	5	1	...	5760

Table 1.1: Generalized Factorial Table for the Subset of Primes

Note, we had to multiply across each row to compute the factorial on that index.

For example, $5!_P = 128 \cdot 9 \cdot 5 \cdot 1 \cdot ... \cdot 5760$

1.3 Generalized Factorials for Specific Cases

Just as we observed above, one can construct the set of generalized factorials by following the procedure provided by Bhargava.

1. **Prime numbers:**

$$k!_S = \{1, 1, 2, 24, 48, 5760, 11520, ...\}$$

2. **Square Numbers:**

$$k!_S = \{1, 1, 12, 360, 20160, ...\} = \frac{(2k)!}{2}$$

3. **Even Integers:**

$$k!_S = \{1, 2, 8, 24, ..., \} = 2^k \times k!$$

4. **Set of Integers of form $an + b$:**

$$k!_S = \{a^k \cdot k!\}_{k=0}^{\infty}$$

5. **Set of integers of form 2^n:**

$$k!_S = \{(2^k - 1) \cdot (2^k - 2) \cdot (2^k - 2^{k-1})\}_{k=0}^{\infty}$$

6. **Natural Numbers:**

$$k!_S = \{1, 1, 2, 6, 24, 120, 720, ..., \} = k!$$

Notice, that even while using this method based on p-orderings and p-sequences, the factorial over the natural numbers remain the same. The author proves that the theorems on the factorials remain true even after the idea of the generalized factorial is applied.

1.4 Theorems Restated

In the beginning of his paper, Bhargava introduces several theorems in number theory that make use of the factorial function and combinatorial identities. Later, he proves that those theorems hold true even when extended to the generalized factorials. These theorems are stated below.

Theorem 1 *For any nonnegative integers k and l, $(k + l)!_s$ is a multiple of $k!_s l!_s$.*

Theorem 2 *Let f be a primitive polynomial of degree k and let $d(S, f) = gcd\{f(a) : a \in S\}$. Then, $d(S, f)$ divides $k!_S$.*

Theorem 3 *The number of polynomial functions from S to $\mathbb{Z}/n\mathbb{Z}$ is given by*

$$\prod_{k=0}^{n-1} \frac{n}{gcd(n, k!_S)}$$

Theorem 4 *Let $a_0, a_1, ...a_n \in S$ be any $n+1$ integers. Then their product of their pairwise differences*

$$\prod_{i<j}(a_i - a_j)$$

is a multiple of $0!_S 1!_S ... n!_S$

Theorem 5 *Let S be a subset of $\mathbb{Z}^n$ (or of R^n, where R is any Dedekind ring). Then for the fixed ordering $M_0, M_1, ...,$ of the monomials $\mathbb{Z}[x_1, ..., x_n]$, a p-ordering of S is a sequence $a_0, a_1, ...$ of elements in S inductively chosen to minimize the highest power of p dividing the determinant*

$$V(a_0, a_1, ..., a_k) = \begin{vmatrix} M_0(a_0) & M_1(a_0) & M_2(a_0) & \dots & M_k(a_0) \\ M_0(a_1) & M_1(a_1) & M_2(a_1) & \dots & M_k(a_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_0(a_k) & M_1(a_k) & M_2(a_k) & \dots & M_k(a_k) \end{vmatrix}$$

The associated p-sequence of S is then given by

$$v_k(S, p) = w_p \frac{V(a_0, a_1, ..., a_n)}{V(a_0, a_1, ..., a_{n-1})})$$

, and the generalized factorial $k!_s$ is

$$k!_s = \prod_p v_k(S, p)$$

[1]

1.5 Posed Questions by the Author

Towards the end of the paper, the author states some questions posed by his audience as well his own questions. Some of the posed questions that interested me prior to this research were as follows:

1. For a subset $S \subset \mathbb{Z}$, is there a natural combinatorial interpretation of $k!_S$?
2. What is the natural combinatorial interpretation for $\binom{n}{k}_S = \dfrac{n!_S}{k!_S(n-k)!_S}$ coefficients?
3. What is the "binomial theorem" for generalized binomial?
4. It appears that the factorials over the subset of primes is the product of the first few Bernoulli numbers. He states that although this can be verified using Von Staudt Theorem, "is there a deeper explanation of this rather striking connection"?

   This question changed the direction of my research process.

   [1]

1.6 Other Papers

Apart from Dr. Bhargava's paper, I read some other papers as well to gain background and insight into this problem. Andrew M. Crabbe's *Generalized Factorial Functions and Binomial Coefficients* relates directly to the initial topic of research. The paper explores the binomial coefficient and how it connects with the generalized factorial. One of theorems is especially fascinating and is described below:

Theorem 6 *For $0 < k < n, n \in \mathbb{N}$, on the generalized Pascal's Triangle for a subset S, the product of six entries surround $\binom{n}{k}_S$ is a perfect square [3].*

$$\binom{n-1}{k-1}_S \cdot \binom{n-1}{k}_S \cdot \binom{n}{k-1}_S \cdot \binom{n}{k+1}_S \cdot \binom{n+1}{k}_S \cdot \binom{n+1}{k+1}_S = a^2, \text{ where } a \in \mathbb{Z} \tag{1.2}$$

Here we can see that using generalized factorials, the author has combined binomial coefficients and square numbers.

Next, Saranya G. Nair and T.N. Shorey utilize the product of consecutive integers to solve other combinatorial problems. They define the product of consecutive integers as follows:

$$\Delta(x, k) = x(x+1) \cdot ... \cdot (x+k-1) \tag{1.3}$$

Instead of utilizing factorials and multiplying with decreasing order, they look at this from another perspective (For example, $\Delta(2,5) = 2 \cdot 3 ... \cdot (2+5-1) = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6)$).

An interesting approach might be to use the Δ function to explore the factorial function. [5]

Although the initial research goal was to explore the relationship between the Binomial Theorem and the Generalized Factorial, interesting questions in Michael R. Pilla's presentation changed the initial goal.

In this presentation, the denominator of certain Taylor series expansions happened to be generalized factorials of certain subsets of the integers. The results from this presentation are stated below.

1.6.1 Summary of Michael R. Pilla's Presentation

1. $(e^x)^m = \sum_{n=0}^{\infty} \frac{m^n}{n!} x^n = 1 + \frac{mx}{1!} + \frac{m^2x^2}{2!} + ...$

2. $(\frac{a}{a-x})^m = 1 + \frac{mx}{a} + \frac{m(m-1)x^2}{2a^2} + \frac{m(m-1)(m-2)x^3}{6a^3} + ... = \sum_{n=0}^{\infty} \frac{P_{a\mathbb{N}+b,n}(m)x^n}{n!_{a\mathbb{N}+b}}$

3. $\cos^m(\sqrt{x}) = 1 - \frac{mx}{2} + \frac{m+3m(m-1)x^2}{24} - ... = \sum_{n=0}^{\infty} \frac{P_{2T,n}(m)x^n}{n!_{2T}}$

4. $2\cos^m(\sqrt{x}) = \sum_0^{\infty} \frac{P_{\mathbb{Z}^2,n}(m)x^n}{n!_{\mathbb{Z}^2}}$

5. $(\frac{-\ln(1-x)}{x})^m = 1 + \frac{mx}{2} + \frac{m...}{24} + \frac{...}{48} = \sum_0^{\infty} \frac{P_{p,n}(m)x^n}{n!_p}$

Summary of n-analogues

Set	Factorial	Analogue
$\mathbb{N}$	$n!_{\mathbb{N}} = n!$	$(e^x)^m$
$a\mathbb{N}+b$	$n!_{a\mathbb{N}+b} = a^n n!$	$(\frac{a}{a-x})^m$
$2T$	$n!_{2T} = (2n)!$	$\cos^m(\sqrt{x})$
$\mathbb{Z}^2$	$n!_{\mathbb{Z}^2} = \frac{(2n)!}{2}$	$2\cos^m(\sqrt{x})$
P	$n!_P = \prod_p p^{\left\lfloor \frac{k-1}{p-1} \right\rfloor + \left\lfloor \frac{k-1}{p(p-1)} \right\rfloor + \left\lfloor \frac{k-1}{p^2(p-1)} \right\rfloor + \ldots}$	$(\frac{-\ln(1-x)}{x})^m$

Table 1.2: Table of Function Analogues to the Generalized Factorial

Chapter 2

Initial Research Questions

2.1 Guiding Questions

Before we had started the research, these were some of the guiding questions that I had proposed.

1. In my earlier research, we derived formulas for taking the differences and sums of any number across any power. Can we find similar formulas if we apply factorial to a number instead of a power? For example do interesting patterns emerge if we observe $6! - 5!$ or $(6^2)! - (5^2)!$

2. If we find a relationship between squares and factorials, can we define combinatorial formulas for permutations and combinations solely through square numbers and the binomial theorem?

3. Dr. Manjul Bhargava utilized a process known as 'p-ordering' to define the generalized factorial. Can we use 'p-ordering' on the binomial theorem for the 'generalized binomial coefficient'?

4. Can the combination of the binomial theorem and generalized factorials be used to solve other problems in a unique way. Examples include "Sum of Two Squares" Problem, or expand on new derivations for other functions like the Riemann Zeta Function. [1]

5. Lastly, before the square number research, I worked on the "Laws of Quadratic Reciprocity." This law is known in mathematics because it has over 200 proofs for the same law. I wonder if any part of our work can be utilized to construct another proof on the Law of Quadratic Reciprocity. [2]

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[1] The Riemann Zeta Function is $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. For example $\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}$.... In 1859, Bernard Riemann published "On the Number of Primes Less Than a Given Magnitude". It contains what is now known as the 'Riemann Hypothesis,' an unproven hypothesis regarding the zeta function which is the basis of several important findings

[2] The Law of Quadratic Reciprocity. gives conditions of the solvability of quadratic equations mod prime numbers. This law is fascinating because of its 240 unique proofs after Gauss published four proofs of his own in the nineteenth century.

2.2 Proposed Methodology

1. The first step is to explore basic patterns. Write out the first few factorials, sums and differences, and products. Then, observe any interesting patterns.

   For example, complete the following lists and note patters:

   (a) $\{0!, 1!, 2!, 3!, 4!, ...\}$

   (b) $\{1! - 0!, 2! - 1!, 3! - 2!, 4! - 3!, ...\}$

   (c) $\{0! + 1!, 1! + 2!, 2! + 3!, 3! + 4!, ...\}$

   (d) $\{(1!)^2 - (0!)^2, (2!)^2 - (1!)^2, (3!)^2 - (2!)^2, (4!)^2 - (3!)^2, ...\}$

   (e) $\{0! \cdot 1!, 1! \cdot 2!, 2! \cdot 3!, 3! \cdot 4!, ...\}$

2. The binomial coefficient itself contains factorials and is defined as follows:

$$\binom{n}{k} = \frac{n(n-1)(n-2)...(n-k+1)}{k!} = \frac{n!}{(n-k)!(k!)}$$

   Crabbe employed the results from Bhargava's paper to connect square numbers and the binomial coefficient.

   (a) First, manipulate Equation (3.1) and utilize our prior research on square numbers and binomial theorems from before to further generalize Crabbe's result.

   (b) Just like Crabbe utilized Bhargava's *results* for the research, we attempt to utilize Bhargava's *process* for our research. This will entail working with p-ordering.

3. The generalized factorial of the square numbers is defined as $(2n)!/2$. Utilize this fact and explore from the frame of the Binomial Theorem.

   Since $\frac{(2n)!}{2} = \{1, 12, 360, ...\}$, compute the following:

   (a) $\{(1!)^2_S - (0!)^2_s, (2!)^2_S - (1!)^2_S, (3!)^2_S - (2!)^2_S, (4!)^2_S - (3!)^2_S, ...\}$, where S is the set of square numbers.

   (b) Compare results with 1 (d).

4. Start with the formulas for permutations and combinations and replace known quantities by the special cases of the binomial theorem we had researched earlier.

Chapter 3

Research Process

3.1 Introduction

With the initial purpose of combining combinatorial identities, the Binomial Theorem, and the Generalized Factorial Function, the first step was to look at concepts from combinatorics. In the process, I came across the following concepts and proceeded to review them. Some of these concepts included:

(i) Ramifications of the Binomial Theorem
(ii) Gaussian Coefficients
(iii) Stirling number of the First Kind
(iv) Stirling number of the Second Kind
(v) 12 Fold Way
(vi) Falling Factorial
(vii) Rising Factorial
(vii) Fibinomial and related identities
(ix) Generating Functions

3.2 Idea

It can be seen from Pilla's work that the sequences of generalized factorials show up in the denominators of several series. We try to find if there is a deeper connection between sequences, series, and generalized factorials.

Some questions concerning calculus naturally comes up. For example, if we integrate or derive the analogues (from Table 1.2), how will the factorial be affected? More importantly, what set did it originally come from?

3.3 Research Question

Given a sequence of numbers, presumably a sequence of generalized factorials for a particular set, can we figure out what set that is?

3.4 A Note on Simulataneous P-Ordering

First, we reduce this problem to the set of sets who have a simulataneous p-ordering. A simulataneous p-ordering occurs when there is a sequence in S that is simulataneous for all primes, p (For example, one can use the $\{1, 2, 3, ...\}$ p-ordering for all primes, so it has a simultaneous p-ordering. On the other hand, if the primes is our set, then each p-ordering is different as we saw in the Examples section).

By Lemma 16 of Bhargava's paper,

$$k!_s = [(a_k - a_0) \cdot (a_k - a_1) \cdot ...(a_k - a_{k-1})] \tag{3.1}$$

[1]

3.5 Motivating Example

Given the set $\{1, 1, 2, 6, 24, 120, 720, ..\}$, can we figure out an algorithm so we can precisely point out that this came from the natural numbers?

1. **Algorithm 1 - Assume Simultaneous P-ordering**

a. Choose the sequence

$S_0 = \{1, 1, 2, 6, 24, 120, 720, ..\}$

b. Generate P-Sequences through Unique Prime Factorization

$p = 2 : \{1, 1, 2, 2, 8, 8, 16, 16, 128, ...\}$

$p = 3 : \{1, 1, 1, 3, 3, 3, 9, ...\}$

$p = 5 : \{1, 1, 1, 1, 1, 5, 5, \}$

c. Pick an index where enough information is given

d. Reversing the p-ordering process, solve the following problem:

$8|(a_5 - a_0) \cdot (a_5 - a_1) \cdot (a_5 - a_2) \cdot (a_5 - a_3) \cdot (a_5 - a_4)$

$5|(a_5 - a_0) \cdot (a_5 - a_1) \cdot (a_5 - a_2) \cdot (a_5 - a_3) \cdot (a_5 - a_4)$

$3|(a_5 - a_0) \cdot (a_5 - a_1) \cdot (a_5 - a_2) \cdot (a_5 - a_3) \cdot (a_5 - a_4)$

Since simultaneous p-ordering is assumed, the variables are the same for all primes.

In the example, at index, we see that the p-sequence for 2,3, and 5 is 8,3, and 5 respectively.

e. Let $d_0, d_1, d_2, d_3, d_4, d_5$ represent the pairwise differences. Then, the following problem can be stated as follows:

$8|d_0 \cdot d_1 \cdot d_2 \cdot d_3 \cdot d_4$

$5|d_0 \cdot d_1 \cdot d_2 \cdot d_3 \cdot d_4$

$3|d_0 \cdot d_1 \cdot d_2 \cdot d_3 \cdot d_4$

As it turns out, the solution to our could not be found using this algorithm as of yet.

2. An alternative way has been utilized to solve this sub-problem.

a. Choose the sequence $S_0 = \{1, 1, 2, 6, 24, 120, 720, ...\}$

b. Generate p-sequences for each prime as shown above.

c. Let $a_0 = 1$ by default. This will not change our results, just the initial condition. The difference between each factorial is going to be the same, so we can rescale it accordingly.

d. Iterate through Bhargava's scheme of p-ordering and p-sequence with $a_0 = 1$. Since we are assuming simulatenous p-ordering, we can use Lemma 3.1 to show that at each step the minimum power is in-fact the generalized factorial.

i. Choose the (minimum) a_1 such that $1|(a_1 - 1) \Rightarrow 1 = a_1 - 1 \Rightarrow a_1 = 2$.

ii. Choose the (minimum) a_2 such that $2|(a_2 - a_0)(a_2 - a_1) \Rightarrow 2 = (a_2 - a_0)(a_2 - a_1)$ and that it has not been repeated before.

$$
\begin{aligned}
2 &= (a_2 - 1)(a_2 - 2) \\
2 &= (a_2^2 - 3a_2 + 2) \\
0 &= a_2^2 - 3a_2 \\
0 &= a_2(a_2 - 3) \\
a_2 &= 0, 3
\end{aligned}
$$

Choose the a_2 that is increasing in the sequence (in this case, $a_2 = 2$. If the other solution is chosen, it will still generate the correct solution, but may start at a different initial location.

iii. Choose a_3 such that $6|(a_3 - a_2)(a_3 - a_1)(a_3 - a_0) \Rightarrow 6 = (a_3 - a_2)(a_3 - a_1)(a_3 - a_0) \Rightarrow 6 = (a_3 - 3)(a_3 - 2)(a_3 - 1)$

One can observe, that now, we are just solving for one variable at a time with an order that is one higher at every iteration.

Note: This problem (for simulataneous p-ordering) has been **solved** (Credit Jon Bolin). One can employ optimization and computational techniques (like Newton's method) to solve for $\{a_0, a_1, ...a_k\}$.

3.6 Other Sub-Problems

These two algorithms above show sub-problems that were needed to solve the problem of reversing the generalized factorials approach. Some of the other sub-problems we encountered and briefly attempted to solve are the following:

1. **Solve Problem with Simultaneous P-orderings**

   This problem was stated above and has been solved by solving equations iteratively. Are there other methods to solve this problem more efficiently? This sub-problem may be interesting to look from an **optimization** point of view (and compare rate of convergence).

2. **Generalize for Non-Simultaneous P-orderings**

   This is the main focus of this paper. Although we have found one way to solve the reverse generalized factorial problem with the assumption of simultaneous p-ordering, we have not yet extended it to sets with non-simultaneous p-ordering.

3. **Number Divides product of differences mod n**

   We saw the following problems in Algorithm 1.

   $8|d_0 \cdot d_1 \cdot d_2 \cdot d_3 \cdot d_4$

   $5|d_0 \cdot d_1 \cdot d_2 \cdot d_3 \cdot d_4$

   $3|d_0 \cdot d_1 \cdot d_2 \cdot d_3 \cdot d_4$

   In this case, $d_0, d_1, ..., d_4$ are represented as $(a_5 - a_0), (a_5 - a_1), ..., (a_5 - a_4)$ respectively.

   The first sub-problem is to solve this. Another interesting problem that stems out is the following:

   What if $(d_i)_{i=0}^{\infty}$ was replaced with different types of expressions (maybe product instead of difference)?

   This could lead to a whole new range of problems to solve. One example is replacing $d_0, d_1, ..., d_4$ with the following $(a_5 \cdot a_0), (a_5 \cdot a_1), ..., (a_5 \cdot a_4)$.

4. **Fractional Derivatives and Generalized Factorials**

   Dr. McKinney's presentation on Fractional Derivatives inspired this question. If we replace derivatives (from integers) to fractions, then the derivatives have a factor of the Gamma Function and Factorial that occur. It would be nice to see what happens if we replace the Factorial in these expressions with the generalized factorial.

   [11]

5. **Explore Repeated Factorials (Ex: Cubes and Fibonacci)**

   In the subset of cubes, we found that $3! = 4! = 504$. Similarly, for the subset of the Fibonacci numbers, $7! = 8! = 443,520$. This was not seen in the factorials for the set of natural numbers. We were able to explain this observation with how the p-orderings and p-sequences were structured for the cubes and Fibonacci set respectively.

   Yet, this could be explored more deeply with questions such as:

a. In what other sets, do we see a repeated factorial phenomenon?

b. Can we construct sets in such a way that repeated Generalized Factorials are guaranteed?

6. **Explore Behavior and Slower Growth Rate (per say) of Cubes**

   The factorial over the cubes grew slower than the factorial over the squares. Intuition says that it would have grown faster.

   Asymptotic behavior and rate of growth between sets could be one area to explore in the future.

Chapter 4

Actual Work and Results

4.1 Phase II: Mathematica Code

First, we needed to code Bhargava's algorithm for generating p-orderings and p-sequences. Hence, a function in Mathematica was generated.

The code was implemented in a step-by-step manner.

1. First, we picked the set of natural numbers. If the code does not work on the natural numbers (whose factorial we know), it will definitely not work on any subsets.

2. Next, we had to design the structure of the code. Our code contained three for-loops.

a. The first for-loop iterated over the different possibilities of a_i that would minimize the exponent of the difference. This was necessary to find the p-ordering and p-sequence for each step.

b. The second for-loop iterated over the entire subset to generate and store the p-orderings and p-sequences for every index.

c. The third for-loop iterated over a **finite** set of primes (this was done for computational purposes).

3. Next, we printed out the p-orderings and p-sequences and checked by hand if they were correct.

4. The final step was to multiply the p-sequences for each index to generate the generalized factorial.

5. We checked if the above process worked for the natural numbers and then proceeded to create a function that would take any subset of the natural numbers.

The body of the code was separated in three major parts: (1) Setup, (2) Body, and (3) Testing. In the setup phase, several example sets were constructed to test later. In part (2), Bhargava's algorithm corresponding to the p-orderings and p-sequences was programmed. And finally, the function was tested with several sets constructed in the setup stage.

4.2 Summary

Some of the important aspects (and description of code) of this phase of research is summarized below.

1. The code is a function that returns the list of the generalized factorials given a subset and an index for which the code is to be evaluated.

   For example, in the testing section, when we write **Bhargava[Naturals,10]**, it means to return the 10! for the set of natural numbers (which are defined in the setup section.

2. Initially, we coded examples from Section 1.3 (Primes, squares, evens, naturals, etc.) and checked if our solutions matched those results.

3. In order to verify if our code was working, I programmed the closed form of a subset with simultaneous p-ordering. In the testing section, 'ActualFactorial,' 'ActualSquares,' and 'ActualEvens' signify the results of the closed form to check our results.

4. The code works successfully only for certain indices. The function is looped over a **finite set of prime numbers** because of computational time (in this case, only the first 20 prime numbers are chosen).

   For example, for the naturals, the code runs successfully up to 19! and for 10! for the set of squares.

   Note: Jon Bolin coded Bhargava Factorials in python and created dynamic list, which can verify our solutions more efficiently.

Chapter 5

Phase III: Proposed Original Algorithm

In the previous chapter, we introduced the problem of reversing the Generalized Factorial for simultaneous p-ordering by solving equations at each iteration.

We now attempt to solve the problem for non-simultaneous p-ordering. In order to accomplish this, we propose a new algorithm and show that it works for sets with simultaneous p-ordering. The difference between this algorithm with the other two proposed earlier is that we do not rule out the case for non-simultaneous p-ordering (although we do not find it explicitly).

5.1 Conjecture/Algorithm

i. Let $\{a_n\}$ be any sequence (for now, take a sequence that we know something about).

ii. Divide the sequence by the factorials on the set of the natural numbers.

iii. Call this new set D.

iv. For each $d \in D$, factor each $d_i = p_i \cdot k_i$ where k_i represents the gaps between successive elements in the original set and p_i represents the 'leftover" unique-prime factorization.

5.2 Examples

Choose the factorials defined on the set of square numbers and work backwards to verify that we, in fact, do get the square numbers back.

5.2.1 Squares

i. $\{a_n\} = \{1, 1, 12, 360, 20160, 1814400, 239500800, ...\}$

ii. Divide each element of $\{a_n\}$ by each element of $n!_{\mathbb{Z}}$. (Ex: 1/1, 1/1, 12/2, 360/6, 20160/24,etc.)

iii. $D = \{1, 1, 6, 60, 840, 15120, 332640., ...\}$

iv. Write the Unique Prime Factorization (UPF) of each element of D.

$$
\begin{aligned}
1 &= 2^0 = \mathbf{1} \cdot 2^0 \\
6 &= 2^1 \cdot 3^1 = \mathbf{3} \cdot 2 \\
60 &= 2^2 \cdot 3^1 \cdot 5^1 = \mathbf{5} \cdot 2^2 \cdot 3 \\
840 &= 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 = \mathbf{7} \cdot 2^3 \cdot 3 \cdot 5 \\
15120 &= 2^4 \cdot 3^3 \cdot 5^1 \cdot 7^1 = \mathbf{9}(3^2) \cdot 2^4 \cdot 3 \cdot 5 \cdot 7 \\
332640 &= 2^5 \cdot 3^3 \cdot 5^1 \cdot 7^1 \cdot 11^1 = \mathbf{11} \cdot 2^5 \cdot 3^3 \cdot 5 \cdot 7 \\
8648640 &= 2^6 \cdot 3^3 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1 = \mathbf{13} \cdot 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \\
&= 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 = \mathbf{15}(3^1 \cdot 5^1) \cdot 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13
\end{aligned}
$$

Note, that the bolded integers are precisely the integer differences between consecutive square numbers. Hence, given an initial condition (in this case, $0^2 = 0$), we have found the initial set of square numbers, which was our goal.

v. $S = \{0, 1, 4, 9, 16, 25, 36, 49, 64, ...\}$ Note, how we factor the UPF is, at this moment, arbitrary. The bolded integers are the factors that changed from one step to the next **BUT** we chose to ignore the new power of 2 was added everytime. Why did we ignore the power of 2, I am not quite sure right now.

5.2.2 (Positive) Even Integers

i. $\{a_n\} = \{1, 2, 8, 48, 384, 3840, 46080, 645120, ...\}$

ii. $n!_{/Z} = \{1, 1, 2, 6, 24, 120, 720, 5040, ...\}$

iii. $D = \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...\}$

iv. UPF listed below:

$$
\begin{aligned}
1 &= 2^0 = 1 \cdot 2^0 \\
2 &= 2^1 = \mathbf{2} \cdot 2^0 \\
4 &= 2^2 = \mathbf{2} \cdot 2^1 \\
8 &= 2^3 = \mathbf{2} \cdot 2^2 \\
16 &= 2^4 = \mathbf{2} \cdot 2^3
\end{aligned}
$$

v. $S = \{0, 2, 4, 6, 8, 10, ..., \}$

Notice, in this case we did not ignore the new power of 2.

5.2.3 Integers similar to 2^n

i. $\{a_n\} = \{1, 1, 6, 168, 20160, 9999360, 20158709760, \}$

ii. $n!_{/Z} = \{1, 1, 2, 6, 24, 120, 720, 5040, ...\}$

iii. $D = \{1, 1, 3, 28, 840, 83328, ...,\}$

iv. UPF

$$
\begin{aligned}
1 &= 2^0 = 2^0 \cdot (\mathbf{2} - 1) \\
3 &= 2^0 \cdot 3^1 = 2 \cdot (\mathbf{4} - 1) \\
28 &= 2^2 \cdot 7^1 = 2^2 \cdot (\mathbf{8} - 1) \\
840 &= 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 2^3 \cdot 7 \cdot (\mathbf{16} - 1) \\
83328 &= 2^7 \cdot 3^1 \cdot 7^1 \cdot 31^1 = 2^7 \cdot 3 \cdot 7 \cdot (\mathbf{32} - 1) \\
&= 2^{11} \cdot 3^2 \cdot 7^2 \cdot 31 = 2^{11} \cdot 31 \cdot (\mathbf{64} - 1) \\
&= 2^{17} \cdot 3^2 \cdot 7 \cdot 31 \cdot 127 = 2^{17} \cdot 3^2 \cdot 7 \cdot 31 \cdot (\mathbf{128} - 1)
\end{aligned}
$$

v. $S = \{1, 2, 4, 8, 16, 32, 64, 128, ...\}$ and the gaps are $2, 4, 8, 16, 32, 64, ...$ respectively.

Notice, we had to embellish one of the prime factors by adding (-1) to get the gap that we wanted. Why is that the case? Not sure.

5.2.4 Trivial Case: Integers

i. $\{a_n\} = \{1, 1, 2, 6, 24, 120, 720, 5040, ...\}$

ii. $n!_{/Z} = \{1, 1, 2, 6, 24, 120, 720, 5040, ...\}$

iii. $D = \{1, 1, 1, 1, 1, ...\}$

iv. Each element of D is just multiplied by 1 from the element before (since each element is the same). Hence, the gap is 1.

v. $S = \{1, 2, 3, 4, 5, ...,\}$

5.2.5 Try Prime Numbers

i $\{a_n\} = \{1, 1, 2, 24, 48, 5760, 11520, 2903040, 5806080, ...\}$

ii $n!_{/Z} = \{1, 1, 2, 6, 24, 120, 720, 5040, ...\}$

iii $D = \{1, 1, 1, 4, 2, 48, 16, 576, 144, 3840, 768, ...\}$

Note, elements of D are not monotonic. Hence, divide into 2 sets, D_1 (comprised of even indices) and D_2 (comprised of odd indices).

$D_1 = \{1, 1, 2, 16, 144, 768, 19968, 552960, ...\}$

$D_2\{1, 4, 48, 576, 684, ...\}$

iv. UPF of D_1

$$
\begin{aligned}
1 &= 2^0 = 1 \\
2 &= 2^1 = \mathbf{2} \cdot 1 \\
16 &= 2^4 = \mathbf{8} \cdot 2 \\
144 &= 2^4 \cdot 3^2 = \mathbf{9} \cdot 2^4 \\
768 &= 2^8 \cdot 3^1 = \mathbf{16} \cdot 2^4 \cdot 3 \\
19968 &= 2^9 \cdot 3^1 \cdot 13^1 = \mathbf{26} \cdot 2^8 \cdot 3 \\
552960 &= 2^{12} \cdot 3^3 \cdot 5^1 = \mathbf{360} \cdot 2^9 \cdot 3 \\
&= 2^{14} \cdot 3^4 \cdot 5^1 \cdot 17^1 \cdot 19^1 = 2 \\
2^{17} \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot 19^1 \cdot 23^1 \cdot 29^1
\end{aligned}
$$

UPF of D_2

$$
\begin{aligned}
1 &= 2^0 = 1 \\
4 &= 2^2 = \mathbf{4} \\
48 &= 2^4 \cdot 3^1 = \mathbf{12} \cdot 2^2 \\
576 &= 2^6 \cdot 3^2 = \mathbf{12} \cdot 2^4 \cdot 3 \\
3840 &= 2^8 \cdot 3^1 \cdot 5^1 = \mathbf{20} \cdot 3 \\
&= 2^{10} \cdot 3^2 = \mathbf{12} \cdot 2^8 \cdot 3 \\
&= 2^{12} \cdot 3^3 \cdot 5^1 \cdot 7^1 = \mathbf{84} \cdot 2^{10} \cdot 3 \cdot 5 \\
&= 2^{14} \cdot 3^3 \cdot 17^1 = \mathbf{102} \cdot 2^{12} \cdot 3^2
\end{aligned}
$$

The gaps are as follows: $\{1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 4, 2, 6, 2, 6\}$

We have not yet figured out how (or if) this relates to the gaps between the primes, but one approach to attempt was to write the prime numbers, look at the differences, and see how that related to the UPF of D_1 and D_2.

5.3 Significance

If this algorithm could help out reverse the generalized factorials to its specific subsets, then several applications are possible.

1. Since some of the generalized factorials relate to the denominators of Taylor expansions, we can apply calculus on it. This includes taking the derivative and integral of the Taylor series expansion, using the new coefficients, and seeing what original subset it came from.

2. Sequences and series play an important role in real analysis. If a more generalized version of this algorithm was implemented, then this may become a tool to analyze any sequence. This may help in discussing rate of convergence, conditional versus absolute convergence, and may be extended to analytic

functions as well (We have so far seen Generalized Factorials, but factorials can be extended with the Gamma function to negative numbers. If such a general form could be found, then complex analysis may be involved in this process as well).

5.4 Other Patterns Found

1. The generalized factorial of the prime subset has an interesting pattern. The factorials are as follows:

   $\{1, 1, 2, 24, 48, 5760, 11520, 5806080, 1393459200, ...\}$

   It can be seen that going from odd to even indices, the generalized factorial always has a factor of 2 (For example, $2 = 2\cdot 1, 48 = 2\cdot 24, 11520 = 2\cdot 5760, ...$).

   On the other hand, going from even to odd indices, the generalized factorial has a factor of 12 ($24 = 12\cdot 2, 5760 = 120\cdot 48 = (12\cdot 10)\cdot 48, 5806080 = 252\cdot 11520 = (12 \cdot 21) \cdot 11520, ..$). The multiple has to do with the new p-sequence that has been multiplied for that particular index.

2. As mentioned in the Proposed methodology, we did some basic algebra and found some interesting patterns at the beginning. One of the interesting patterns, that relate factorials to the square numbers is shown below.

$$\begin{gathered}
1! - 0! = 1 = 0^2 \cdot 1 \\
2! - 1! = 1 = 1^2 \cdot 1 \\
3! - 2! = 4 = 2^2 \cdot 1 = 2^2 \cdot 0! \\
4! - 3! = 18 = 3^2 \cdot 2 = 3^2 \cdot 1! \\
5! - 4! = 96 = 4^2 \cdot 6 = 4^2 \cdot 2! \\
6! - 5! = 600 = 5^2 \cdot 24 = 5^2 \cdot 3! \\
7! - 6! = 1320 = 62 \cdot 120 = 6^2 \cdot 5! \\
8! - 7! = 35280 = 7^2 \cdot 720 = 7^2 \cdot 6!
\end{gathered}$$

The pattern is explaained below:

$$\begin{aligned}
(x+1)! - x! &= (x+1)x! - x! \\
&= x![x+1-1] \\
&= x!(x) \\
&= xx! \\
&= x(x)(x-1)! \\
&= x^2 \cdot (x-1)!
\end{aligned}$$

This was an interesting pattern that started our investigations in this project. We encountered other patterns periodically but these two were the major ones we have not yet covered.

Chapter 6

Conclusion

6.1 What did I learn?

This project started as an attempt to combine my earlier research relating to the Binomial Theorem and relating it to Dr. Bhargava's Generalized Factorial paper. Although we initially took this route, our end result was in a completely different direction.

I learned more about the research process and how to efficiently spend time to tackle certain problems. For the first half of our research process, I was reading background material, papers related to this topic, and library books on combinatorics. In order to start doing original research, I used the proposed methodology to begin finding patterns. Later, in Phase II, we coded the p-ordering and p-sequences in Mathematica. This provided us the tool to drive in a different direction.

Earlier, we only had resources from Bhargava's and Crabbe's paper on certain results. Since our code worked for any subset (up to a certain degree of accuracy), we could generate generalized factorials for subsets that we had not seen before. Hence, we explored the cubes, fourths, Fibonacci numbers, etc. With these results, we had further data to analyze, which gave us several more patterns to uncover.

Lastly, Pilla's presentation drove our work in Phase III. I am intrigued by power series, and the relationship between Taylor series expansions and the generalized factorial became our focus. With Taylor series, one can manipulate the coefficients, take the derivative and integrals, etc. This has implications with the radius of convergence, conditional versus absolute convergence, etc. Taylor series can be used to evaluate limits and can make conclusions of the smoothness of a given function. Maybe, this may lead to interesting cross overs between number theory, calculus, real analysis, and maybe even functional analysis.

Throughout this process, I learned about concepts from number theory and combinatorics. Furthermore, I learned that even though some patterns look interesting, they might not have deep conclusions. On the other hand, innocuous looking patterns may lead to a significant result. Lastly, especially towards the end, when we were trying to find the reverse generalized factorial for the primes, I found numerous patterns and went into details. Although I was going really deep, the result did not come as of yet. This was frustrating but made me realize, this was all part of the process. I will continue looking at this, and hopefully solve this problem in the near future.

6.2 Correspondence

Last year, I emailed Dr. Bhargava to see if any progress has been made on the posed question relating the Generalized Factorial to the Binomial Theorem. He graciously responded and conveyed that no significant contribution has been found but is curious to see if we find something.

6.3 About the Main Referenced Author - Dr. Manjul Bhargava

- Paper Published in 2000
- Since worked on Higher Composition Laws, 15 and 290 Theorems, and Average Rank of Elliptic Curves
- Fields Medal Recipient in 2014
- Doctoral Advisor: Andrew Wiles
- Professor at Princeton, Leiden University, and adjunct professor in several others.
- Musician (Tabla Player)

Acknowledgements

I want to thank my mom, dad, and my brother for everything. This would not have been possible without their support and encouragement.

I am extremely grateful to the University of Tulsa and the Department of Mathematics, especially to Dr. O'Neil, who has mentored and guided me for the past four years.

Special thanks to Jon Bolin, who thought about these problems with me, helped me debug code, and generated his own original code to help with this process. Also thanks to Brian Luczak, Alex Hunn, and Vandit Trivedi for listening to me along the way and encouraging me to continue.

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