Three-Parameter Bayesian Estimation Problem

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Two-Parameter Beta Distribution

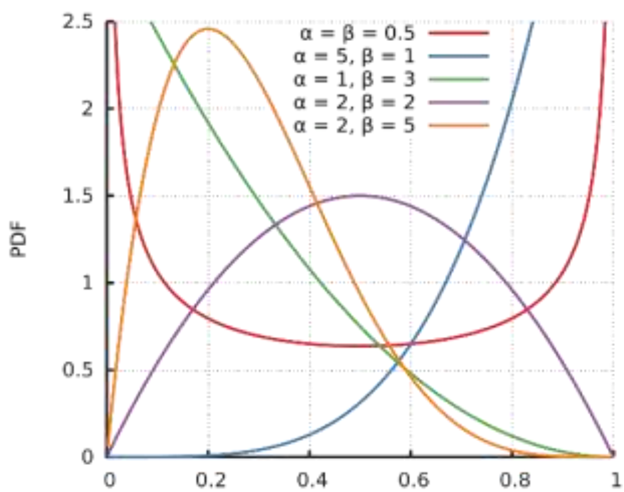
$\mathrm{Beta}(\alpha, \beta)$

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}$$

where $0 < x < 1$,

$$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

$\alpha > 0, \beta > 0$

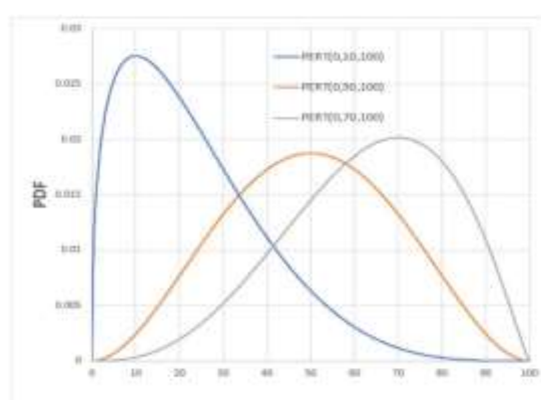


Four-Parameter Beta Distribution $\text{Beta}(\alpha, \beta, a, c)$

- Derive from $\text{Beta}(\alpha, \beta)$ by mapping $[0,1]$ to $[a, c]$.
- $f(x) = \frac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{\text{B}(\alpha,\beta)(c-a)^{\alpha+\beta-1}}$
  where $a < x < c$, $a < c$, $\alpha > 0$, $\beta > 0$

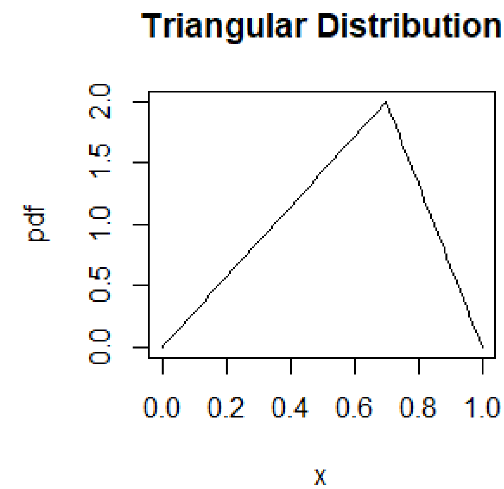
$\text{PERT}(a, b, c)$ Distribution

- support = $[a, c]$
- $a < b < c$
- b is the mode
- $\mu = \frac{a+4b+c}{6}$
- $\text{Beta}\left(\frac{4b+c-5a}{c-a}, \frac{5c-a-4b}{c-a}, a, c\right)$
- Program Evaluation and Review Technique by CE Clark 1962



Triangular Distribution $\text{Triang}(a, b, c)$

- support = $[a, c]$
- $a < b < c$
- b is the mode
- simple replacement for PERT



Consulting Problem

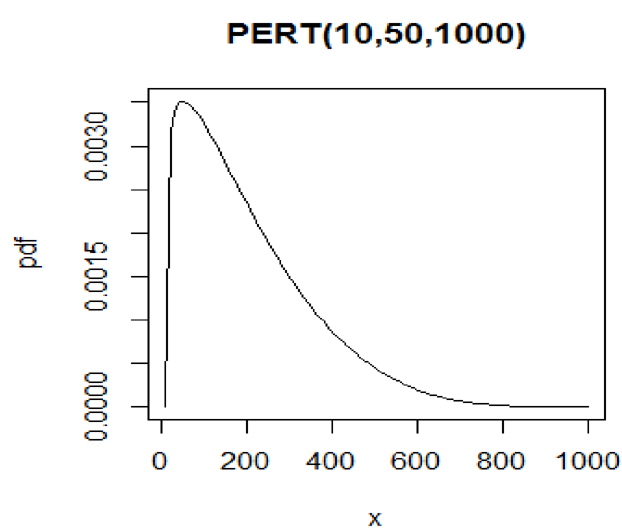
- Assume $(A, B, C) = (min, mode, max)$ is independent
- $A \sim \text{Triang}(6,10,14), B \sim \text{Triang}(30,50,70),$
  $C \sim \text{Triang}(900,1000,1100)$
- $A = 10 \pm 4, B = 50 \pm 20, C = 1000 \pm 100$
- Observe a single sample from $\text{PERT}(a, b, c)$ with pdf $f(x|a, b, c)$.
- Update the parameters a, b, c for the sample $x = 8$ using Bayesian techniques.
- These numbers were supplied by an analyst at a large corporation.

Prior Probability Density Function

- $h(p)$ where $p = (a, b, c)$ for the parameters (A, B, C)
- R library mc2d (Tools for Two-Dimensional Monte-Carlo Simulations)
- Product of three dtriang(x , min, mode, max)
- support = $[6,14] \times [30,70] \times [900,1100] \subset \mathbb{R}^3$.

Sample Probability Density Function $f(x, 10,50,1000)$

- R function dpert(x, min, mode, max)
- mc2d library

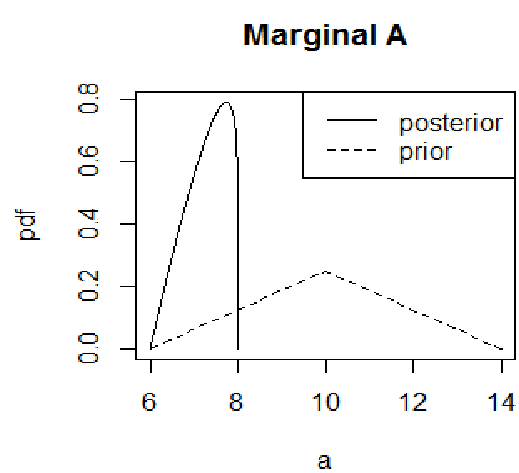


Posterior Probability Distribution for (A, B, C)

- proportional to the joint distribution of (X, A, B, C) when $X = 8$:
  $$joint(p) = f(x = 8|a, b, c)h(a, b, c)$$
- $\int_6^{14} \int_{30}^{70} \int_{900}^{1100} joint \approx 0.0002406715$
- R function adaptIntegrate in cubature library
- posterior pdf $k(a, b, c|x = 8) = joint(p)/0.0002406715$

Posterior Marginal Distribution of A

- $f_A(a) =$
  $$\int_{30}^{70} \int_{900}^{1100} k(a, b, c|x = 8)\, dc\, db$$
- Approximated using adaptIntegrate

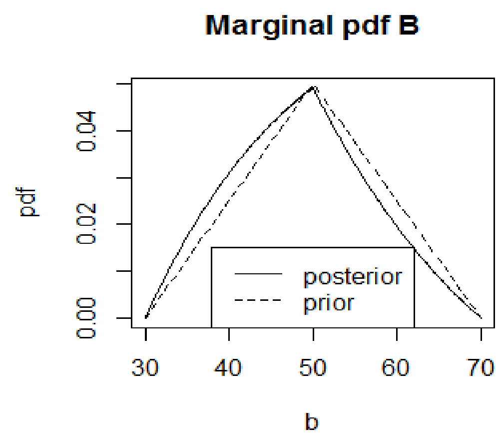


Posterior Marginal Distribution of B

- $f_B(a) =$

$$\int_{6}^{14}\int_{900}^{1100} k(a,b,c|x=8)\,dc\,da$$

- Approximated using adaptIntegrate
- Took 23 seconds for lapply to make pdf vector.

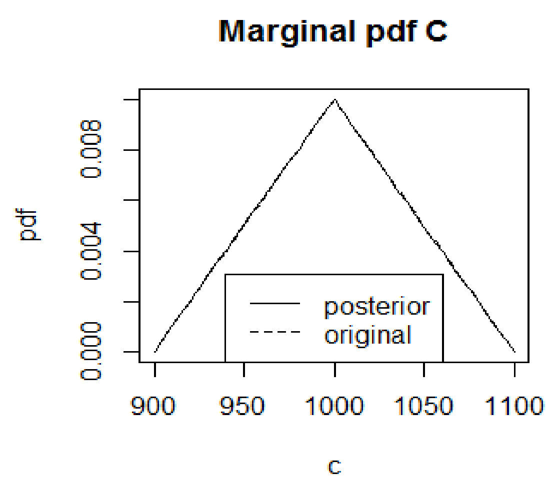


Posterior Marginal Distribution of C

- $f_C(a) =$

$$\int_{6}^{14}\int_{30}^{70} k(a,b,c|x=8)\,db\,da$$

- Approximated using adaptIntegrate



Square Loss Function Estimates

- Minimize $E[(\Theta - \theta_{est})^2 | X = 8]$ to estimate the parameter θ.
- $a_{est} = E[A|x = 8] = \int_6^{14} \int_{30}^{70} \int_{900}^{1100} a\, k(a, b, c|x = 8)\, dc\, db\, da \approx 7.27$
- $b_{est} = E[B|x = 8] = \int_6^{14} \int_{30}^{70} \int_{900}^{1100} b\, k(a, b, c|x = 8)\, dc\, db\, da \approx 48.5$
- $c_{est} = E[C|x = 8] = \int_6^{14} \int_{30}^{70} \int_{900}^{1100} c\, k(a, b, c|x = 8)\, dc\, db\, da \approx 1000$

Absolute-Value Loss Estimates

- Minimize $E[|\Theta - \theta_{est}||X = 8]$ to estimate the parameter θ.
- $\frac{1}{2} = P\left[A < \mathrm{M}_{A|X=8}\right] = \int_6^{a_{est}} \int_{30}^{70} \int_{900}^{1100} k(a, b, c|x = 8)\, dc\, db\, da \rightarrow a_{est} \approx 7.33$
- Similarly, $b_{est} = \mathrm{M}_{B|X=8} \approx 48.4$
- Similarly, $c_{est} = \mathrm{M}_{C|X=8} \approx 1000$

Conclusion

- prior parameters $(a, b, c) = (10, 50,1000)$
- square loss function estimates $(7.27,48.5,1000)$
- absolute-value loss function estimates $(7.33,48.4,1000)$
- It was less convenient to use the absolute-value loss function and did not make much difference.
- The greatest effect of the sample $x = 8$ was on a and the least effect was on c.

References

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