Expectation and the Riemann Zeta Function

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ABSTRACT

When supplying the student with examples of calculating expectation for a random variable X, it is important to point out that $E(X)$ may not be finite. In this classroom note, we explore an example where the expectation is finite but the exact value is not known. This leads to an interesting connection and result involving the Riemann Zeta function.

For the past three years during the spring semester, the author has taught a course at Harding University entitled MATH 318; Probability. During the spring semester of 2018, inspiration was gained from the following example (see page 209 in [2]). In this paper, we use the notation and terminology of [2] which is the textbook used in MATH 318.

Example 1. Let $\boldsymbol{X}$ be a random variable with probability function given by

$$f(x) = \begin{cases} \dfrac{1}{x(x+1)} & if\ x = 1,2,3,\dots \\ 0 & \text{otherwise.} \end{cases}$$

By definition

$$E(X) = \sum_{x=1}^{\infty} x \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \frac{1}{(x+1)} = \infty,$$

therefore X is a random variable with expectation that is not finite. ■

A slight change to the probability function in example 1 allows us in example 2 to make connections to a famous sum due to Euler (see chapter 9 in [3]).

Example 2. Let X be a random variable with probability function given by

$$f(x) = \begin{cases} \frac{c}{x^2} & if\ x = 1,2,3,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

As in example 1, $\boldsymbol{E}(\boldsymbol{X})$ is not finite. Perhaps a more interesting question for the student to investigate is what value of $\boldsymbol{c}$ is necessary so that $\boldsymbol{f}$ is a valid probability function? This of course is not an easy calculation. The reader will likely recognize this as being connected to a problem first solved by Leonhard Euler in 1734. We invite the reader to the exposition in chapter 9 of Dunham's book [3] for a look at the history as well as the mathematics behind the solution to this problem. Euler proved that

$$\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6},$$

therefore the value of $c = \frac{6}{\pi^2}$. ■

Next we give the following definition which can be found on page 187 of [1].

Definition. The Riemann zeta function is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \text{ for } Re\ z > 1.$$

One of the most famous unsolved problems in all of mathematics is the location of the zeros of the zeta function [1]. The reader can readily see that Euler proved that $\zeta(2) = \frac{\pi^2}{6}$. In fact, exact values of the zeta function for input equal to an even integer can be computed using Euler's techniques expounded upon in chapter 9 of [3]. For instance, Euler proved that $\zeta(4) = \frac{\pi^4}{90}$. We are ready to make a connection to yet another famous unsolved problem. The exact value of the zeta function for an input equal to an odd integer is unknown. Regarding $\zeta(3)$, Dunham says in [3], "it is easy to conjecture that the sum in question is equal to $\frac{p}{q}\pi^3$." While we do not solve the problem of calculating an exact value of $\zeta(3)$, we obtain an interesting estimate involving π^3 which uses the technique of expectation. First we give our last example of a random variable with a unique expected value.

Example 3. Let X be a random variable with probability function given by

$$f(x) = \begin{cases} \frac{c}{x^4} & if\ x = 1,2,3,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

We see that $c = \frac{1}{\zeta(4)} = \frac{90}{\pi^4}$. Therefore, if we compute the expected value for X, we get

$$E(X) = \frac{90}{\pi^4}\sum_{x=1}^{\infty} x\frac{1}{x^4} = \frac{90}{\pi^4}\sum_{x=1}^{\infty}\frac{1}{x^3} = \frac{90}{\pi^4}\zeta(3). \tag{1}$$

This gives the student an example where $E(X)$ is finite but the exact value is unknown. ■

We are ready to state and prove the main result of this paper.

Proposition. $\zeta(3) \leq \frac{\sqrt{15}}{90}\pi^3$.

Proof. Let X be the random variable of example 3. Note that

$$E(X^2) = \frac{90}{\pi^4}\sum_{x=1}^{\infty} x^2\frac{1}{x^4} = \frac{90}{\pi^4}\sum_{x=1}^{\infty}\frac{1}{x^2} = \frac{90}{\pi^4}\zeta(2) = \frac{90}{\pi^4}\times\frac{\pi^2}{6} = \frac{15}{\pi^2}.$$

We now use the fact that $Var(X) = E(X^2) - [E(X)]^2$. Solving for $E(X)$ yields

$$E(X) = \sqrt{E(X^2) - Var(X)} \leq \sqrt{E(X^2)}$$

Since $Var(X) \geq 0$ (see Theorem 4.3.2 in [2]). Using (1) in example 3 and the value for $E(X^2)$,

we get $\frac{90}{\pi^4}\zeta(3) \leq \sqrt{\frac{15}{\pi^2}}$ or $\zeta(3) \leq \frac{\sqrt{15}}{90}\pi^3$. ■

In conclusion, using techniques of random variables and expectation, we are able to provide a unique example of a random variable that has a finite expectation with unknown exact value. Moreover, the reader can verify that we have the added bonus that the estimate we obtain for $\zeta(3)$ is equal to the geometric mean of $\zeta(2)$ and $\zeta(4)$!

REFERENCES

1. Conway J. B., *Functions of One Complex Variable*, Springer-Verlag, New York, 1978.
2. DeGroot M. H., M. J. Schervish, *Probability and Statistics 4th ed.*, Addison-Wesley, 2012.
3. Dunham W., *Journey through Genius: The Great Theorems of Mathematics*, Penguin Books, London, 1990.