Partisansip's Effect on the United State's Electoral System

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June 19, 2018

This paper will primarily discuss topics related to voting power, and a metric used to determine a voter's influence: the Banzhaf-Penrose Power Index. This metric was originally introduced by Lionel Penrose in 1946, but was further studied and made famous by an attorney by the name of John Banzhaf III in 1965. In this paper, this metric will be referred to as the Banzhaf Power Index (abbrev. BPI).

In 1964, Attorney John Banzhaf III wrote a paper titled *Weighted Voting Doesn't Work: A Mathematical Analysis.* The paper specifically cited a situation in Nassau County, New York. The Nassau County Board of Supervisors used a weighted voting system, allocating a fixed number of votes to representatives from every region in the county based on population. Banzhaf's thesis in this paper asserted that assigning weighted votes approximately proportionate to the region's population size was insufficient. The argument was compelling, and Nassau County overhauled their voting system in 1993 after several decades of legal battles.

A question frequently asked with respect to the U.S. voting system is "how much does my vote matter?" A sufficient response to this question is much more involved than it initially appears. It's tempting to jump right to the matter of how much *weight* a player has in a voting system, but that does not give a realistic view into how much influence a voting member has on the overall outcome. To answer the question regarding influence, we will observe a statistic known as the Banzhaf Power Index (BPI) that gives insight into a notably more valuable factor: *Voting Power*. Note that the Banzhaf Power Index is not the only statistic that measures voting power, but the BPI and probabilistic measures will be the focus of this article.

Voting Power

A common metric used for measuring the potential influence of an undecided voter in a *weighted* voting system is the Banzhaf Power Index. The BPI of any player in a voting system is calculated using the proportion of the number of times this specific voting member (member A, for instance) is "critical" to the number of times *all* players are critical:

$$BPI(Member\ A) = \frac{Sum\ of\ instances\ member\ A\ is\ critical}{Sum\ of\ instances\ all\ members\ are\ critical}\ . \tag{1}$$

In this context, a "critical" vote is one that overturns the overall decision in a winning coalition (a combination of votes that wins the majority). Put simply, the Banzhaf Power Index provides the probability that a voter would play a "critical" role. Note that the Banzhaf Power Index is used to give valuable insight into voting influence in *weighted* voting systems. In unweighted voting systems (i.e., all members have an equally weighted vote), the BPI for each voter will be $\frac{1}{N}$, taking N as the number of participating voting members.

Before we can look at examples regarding weighted voting systems, there are a couple mathematical derivations we must observe to assist us in gaining further insight into the mathematical intricacies of voting power. These are the same tools Professor Lionel Penrose used in the original paper regarding voting power using the Banzhaf Power Index.

Tools

Stirling's Approximation A mathematician by the name of James Stirling derived a formula that gives a rather accurate estimation for *n!* (n factorial). Recall the calculation for factorials:

$$n! = n \cdot (n-1) \cdot (n-2) \ldots (2) \cdot (1)\ .$$

Stirling comes to the conclusion that n-factorial may be approximated by the following:

$$n! \approx e^{-n} n^n \sqrt{2\pi n}\ \ \forall\ n \in \mathbb{N}\ . \tag{2}$$

Note that this is simply an approximation, and does not give exact results for small values of n. However, the approximation works well for values of n in most practical contexts. A brief table is shown below to demonstrate the accuracy of this approximation for several natural numbers.

n	n!	Approx.	% error
3	6	5.8362	2.81
5	120	118.019	1.67
10	3628800	3598695.62	0.83

NOTE: As n increases by a factor of 10, percentage of error decreases by a factor of 10.

Before we dive into an example, let's review binomial coefficients. Recall the calculation for unordered combinations:

$$\binom{a}{b} = \frac{a!}{b! \cdot (a-b)!} \ .$$

Binomial coefficients are used to determine the number of combinations possible, given a population size (a) and a sample size (b). The above equation calculates the "number of unordered combinatorial possibilities when choosing b objects from a population of size a."

Now, let's look at a few situations that shed light onto voting power as it might apply to a voter in the United States. A couple of these examples will explore unweighted voting systems.

Application

Example 1: Consider a situation where there is a population of N voters in a state, where *N+1* is an odd integer. Suppose a person named Bob is a citizen within this state. In this scenario, assume every voter only has two options on voting day, "yea" or "nay." Additionally, every citizen within this state is equally likely to vote either way. What is the probability Bob's vote is critical?

To begin to answer this question, we must first find in what scenario(s) Bob's vote would be critical, only then is obtaining a probability plausible. As covered previously, a vote is critical *if and only if* the vote swings the outcome of the overall decision. Since only a simple majority is necessary to change the overall decision in this scenario, $\frac{N}{2} + 1$ votes are needed to swing the vote in either of the two directions, yea or nay. Hence, Bob's vote would be critical in a number of situations given by

$$\binom{N}{\frac{N}{2}} = \frac{N!}{(\frac{n}{2})!(\frac{n}{2})!} \ .$$

Due to the fact that there are two choices on the ballot, the number of coalitions (i.e., arrangements of voters between "yea" and "nea") is given by 2^{N+1}. Since the Banzhaf Power Index is a probability that a voter is critical, we must divide the number of situations in which the voter is critical by all voting scenarios (given by 2^{N+1}). Additionally, we must multiply the combinatorial fraction by a factor of 2 to account for the fact that there is a symmetric losing coalition for every winning coalition that we observe in which a certain voter is critical (i.e., there are two choices on the ballot).

So, the probability Bob's vote is critical can be obtained by

$$\begin{aligned}\binom{N}{\frac{N}{2}} \cdot \frac{2}{2^{N+1}} &= \frac{N!}{(\frac{n}{2})!(\frac{n}{2})!} \cdot \frac{2}{2^{N+1}} \\ &= \frac{N!}{(\frac{n}{2})!(\frac{n}{2})!} \cdot \frac{1}{2^{N}} \ . \end{aligned} \tag{3}$$

Now, for the sake of cleanliness and readability, suppose we set M, an arbitrary variable, equal to $\frac{N}{2}$:

$$M = \frac{N}{2} \Rightarrow N = 2M \ .$$

Equation (3) may now be rewritten as

$$\binom{2M}{M} \cdot \frac{2}{2^{N+1}} = \frac{(2M)!}{(M)!(M)!} \cdot \frac{1}{2^{2M}} \ . \tag{4}$$

Using Stirling's approximation for factorials, equation (4) can now be rewritten as

$$\binom{2M}{M} \cdot \frac{2}{2^{N+1}} = \frac{e^{-2M} 2M^{2M} \sqrt{2\pi 2M}}{\left(e^{-M} M^{M} \sqrt{2\pi M}\right)\left(e^{-M} M^{M} \sqrt{2\pi M}\right)} \cdot \frac{1}{2^{2M}} \ .$$

After a bit of grueling simplification, we obtain

$$\frac{\sqrt{\pi M}}{\pi M} = \frac{1}{\sqrt{\pi M}} \ .$$

Once we plug $\frac{N}{2}$ back in for M, we obtain the result

$$\begin{aligned}\frac{1}{\sqrt{\pi M}} &= \frac{1}{\sqrt{\frac{\pi N}{2}}} \\ &= \sqrt{\frac{2}{\pi N}} \ . \end{aligned} \tag{5}$$

Example 2: Consider a situation where there is a population of 8 voters in a state. Suppose a man named Bob is a citizen within this state. Assume voters only have two options on the ballot. Let's call them Party A and Party B. Similar to the last example, assume voters are equally likely to vote one way or the other. What is Bob's voting power?

Here, we will outline a few algebraic steps in an example of application. First, refer back to equations (2) and (4).

Now, let's substitute our data into equation (3), assuming $N = 8$:

$$\binom{N}{\frac{N}{2}} \cdot \frac{2}{2^{N+1}} = \frac{8!}{(4)!(4)!} \cdot \frac{1}{2^8} \ . \tag{6}$$

Using Stirling's approximation for factorials, equation (6) can now be rewritten as

$$\begin{aligned}
\frac{8!}{(4)!(4)!} \cdot \frac{1}{2^8} &\approx \frac{e^{-8} 8^8 \sqrt{2\pi 8}}{\left(e^{-4} 4^4 \sqrt{2\pi 4}\right)\left(e^{-4} 4^4 \sqrt{2\pi 4}\right)} \cdot \frac{1}{2^8} \\
&\approx \frac{(e^{-8})(8^8)(2)\sqrt{4\pi}}{(e^{-8})(4^8)(2)(4\pi)} \cdot \frac{1}{2^8} \\
&\approx \frac{(8^8)}{(8^8)\sqrt{4\pi}} \\
&\approx \frac{1}{\sqrt{4\pi}} \ .
\end{aligned}$$

Notice how this result compares to the derivation in the previous example:

$$\begin{aligned}
\sqrt{\frac{2}{\pi N}} &= \sqrt{\frac{2}{\pi 8}} \\
&= \frac{1}{\sqrt{4\pi}} \ .
\end{aligned}$$

Example 3: Suppose we have a state within the united States that has a population of 101 citizens, including Bob. Also, assume all 101 citizens will vote. Now, assume that a census has been conducted, determining the *decisive* votes of the 101 citizens. For this example, assume that all voters will either vote Republican or Democrat. The survey determined there are 20 citizens who will *definitely* vote Democrat, and 30 citizens who will *definitely* vote Republican. Lastly, there are 51 citizens (including Bob) who are not entirely decisive, and could vote either way on voting day. Suppose we wish to calculate Bob's voting power within this state.

Let's define N as the total population, and define u as the number of independent/undecided voters:

$$N = 100$$

$$Democratic\ Voters\ (Count) = 20$$

$$Republican\ Voters\ (Count) = 30\ .$$

Since we are calculating the voting power for the independent voters, we will need to find two things: 1) the number of situations in which Bob's vote is critical and 2) all the number of voting situations. For Bob's vote to be critical, there must be a 50/50 split between Democratic votes and Republican votes, so that Bob will be the decision maker. For that to be the case, 30 of the independent voters must vote democrat, and 20 of the independent voters must vote Republican to obtain this even split. Since the equality

$$\binom{50}{20} = \binom{50}{30}$$

holds, we may use either to calculate the number of situations in which Bob's vote is critical. Note that there are 2^{50} ways in which these undecided voters can vote, which is why we are dividing by that value to calculate an undecided voter's power.

$$\binom{50}{20} \cdot \frac{2}{2^{50+1}} = \frac{(50)!}{(20)!(30)!} \cdot \frac{1}{2^{50}} \ . \tag{7}$$

Again, we will be using Stirling's approximation to simplify equation (8):

$$\frac{(50)!}{(20)!(30)!} \cdot \frac{1}{2^{50}} \approx \frac{e^{-50}50^{50}\sqrt{2\pi 50}}{\left(e^{-20}(20)^{20}\sqrt{2\pi 20}\right)\left(e^{-30}(30)^{30}\sqrt{2\pi 30}\right)} \cdot \frac{1}{2^{50}} \ .$$

We can rewrite this as:

$$\begin{aligned}
&\approx \frac{50^{50}}{(30)^{30}(20)^{20}\sqrt{.48\pi 50}} \cdot \frac{1}{2^{50}} \\
&\approx \frac{50^{50}}{(.6)^{30}(.4)^{20}(50)^{30}(50)^{20}\sqrt{.48\pi 50}} \cdot \frac{1}{2^{50}} \\
&\approx \frac{1}{(.6)^{30}(.4)^{20}\sqrt{.48\pi 50}} \cdot \frac{1}{2^{50}} \ .
\end{aligned}$$

Rewriting decimals as fractions, we obtain:

$$\begin{aligned}
&\approx \frac{1}{\left(\frac{3}{5}\right)^{\frac{3}{5}(50)}\left(\frac{2}{5}\right)^{\frac{2}{5}(50)}\sqrt{(2)(\frac{3}{5})(\frac{2}{5})\pi(50)}} \cdot \frac{1}{2^{50}} \\
&\approx \frac{5^{\frac{3}{5}(50)} \cdot 5^{\frac{2}{5}(50)}}{3^{\frac{3}{5}(50)} \cdot 2^{\frac{2}{5}(50)}\sqrt{(2)(\frac{3}{5})(\frac{2}{5})\pi(50)}} \cdot \frac{1}{2^{50}} \\
&\approx \left(\frac{5}{3}\right)^{\frac{3}{5}(50)} \cdot \left(\frac{5}{2}\right)^{\frac{2}{5}(50)} \cdot \left(\frac{1}{2}\right)^{50} \cdot \frac{1}{\sqrt{(2)(\frac{3}{5})(\frac{2}{5})\pi(50)}} \ .
\end{aligned}$$

When simplifying using Stirling's approximation, this method turns out to have some pretty interesting algebraic properties that allow for useful simplification.

Theorem 1. *Suppose there exists a voting game that consists of two parties, Party A and Party B. The party with a simple majority of N ($\frac{N}{2}+1$ votes) wins the overall vote. Let N denote the number of voting players within said game. Suppose we have a predetermined number of voters within population N that have decided to vote for Party A or Party B. Let u define the number of undecided voters within population N. Let α define the predetermined number of voters for Party A, and let β define predetermined number of voters for Party B. Assume that the number of decided predetermined voters in a system must be less than half of the number of all voters (i.e. α and β must be less than $\frac{N}{2}$). Then, the probability an undecided voter is critical within this game can be calculated using the following:*

$$P_c(u) \approx \begin{cases} 0 & \text{if } \alpha > 0.5 \text{ or } \beta > 0.5 \\ \left[(P_A)^{-P_A} \cdot (P_B)^{-P_B} \cdot \frac{1}{2}\right]^u \cdot \left(\sqrt{2\pi u (P_A)(P_B)}\right)^{-1} & \text{if } \alpha \leq 0.5 \text{ and } \beta \leq 0.5 \end{cases}$$

where

$$P_A = \frac{0.5N - \alpha}{u}$$

$$P_B = \frac{0.5N - \beta}{u} \quad .$$

Proof. Consider the voting game given in Theorem 1, along with its necessary assumptions:

$$P_A = \frac{0.5N - \alpha}{u} = \frac{x}{u}$$

$$P_B = \frac{0.5N - \beta}{u} = \frac{y}{u}$$

$$u = N - \alpha - \beta \ = x + y$$

$$x = 0.5N - \alpha \ = (u)(P_A)$$

$$y = 0.5N - \beta \ = (u)(P_B)$$

.

With x and y defined as the number of voters needed from Parties A and B to split the vote evenly, making an undecided voter critical, we can calculate the power for an undecided voter:

$$\begin{aligned}
P_c(u) &= \binom{u}{x} \cdot \frac{2}{2^{u+1}} = \binom{u}{y} \cdot \frac{2}{2^{u+1}} \\
&= \frac{u!}{(x)!(y)!} \cdot \frac{1}{2^u} \\
&\approx \frac{(e^{-u})(u)^u \sqrt{2\pi u}}{(e^{-x})\,(x)^x\,(e^{-y})\,(y)^y (2\pi)\sqrt{(x)(y)}} \cdot \frac{1}{2^u} \\
&\approx \frac{(u)^{x+y}\sqrt{u}}{(x)^x (y)^y \sqrt{2\pi (x)(y)}} \cdot \frac{1}{2^u} \\
&\approx \left(\frac{u^x}{x^x}\right) \cdot \left(\frac{u^y}{y^y}\right) \cdot \frac{\sqrt{u}}{\sqrt{2\pi (u \cdot P_A)(u \cdot P_B)}} \cdot \frac{1}{2^u} \\
&\approx (P_A)^{-x} \cdot (P_B)^{-y} \cdot \frac{1}{\sqrt{2\pi u (P_A)(P_B)}} \cdot \frac{1}{2^u} \ .
\end{aligned}$$

□

Corollary 1.1. *In a voting system in which there are no predetermined voters,*

$$P_c(u) = \sqrt{\frac{2}{\pi N}} \ .$$

Proof. In a situation in which there are no predetermined voters,

$$u = N$$

$$P_A = \frac{0.5N - a}{u} = \frac{0.5N}{N} = \frac{1}{2}$$

$$P_B = \frac{0.5N - b}{u} = \frac{0.5N}{N} = \frac{1}{2} \ .$$

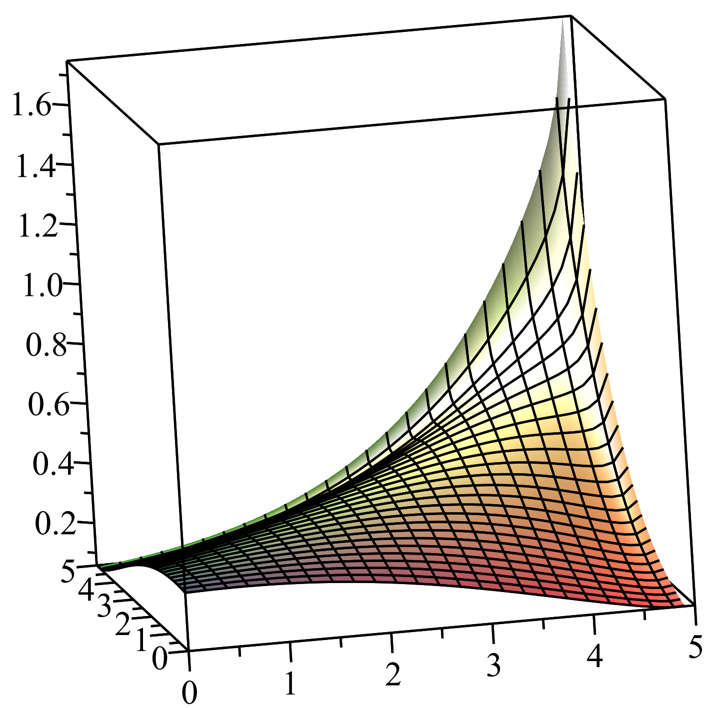
If we plug these into our generalized solution:

$$\begin{aligned}
P_c(u) &= \left(2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} \cdot \frac{1}{2}\right)^N \cdot \frac{1}{\sqrt{(2)(\frac{1}{2})(\frac{1}{2})\pi N}} \\
&= (1)^N \cdot \left[\frac{1}{\sqrt{\frac{\pi N}{2}}}\right] \\
&= \sqrt{\frac{2}{\pi N}} \ .
\end{aligned}$$

□

This proof helps us gain a clearer view on the concept of voting power, especially if we make a few observations about the function. First, note the purpose of the function: to calculate voting power for an *undecided* voter. Voting power with regard to a predetermined voter renders meaningless, because they have preallocated their vote. A necessary assumption for calculating voting power is that the voting member in question has the ability to go *either* way on voting day. Next, lets observe how the values of voting power change as a function of α and β, the number of predetermined voters for a party within a population N. We can do this by graphing a surface of the generalized voting power as a function of α and β. The surface for $N = 10$ is displayed below.

N = 10



First, notice when there are few predetermined voters, the potential influence for an undecided voter is significantly larger compared to any other potential voting scenario. Also, notice as predetermined voter imbalance increases, potential influence for undecided voters decreases.

Now, let's take a look at some of the results of this generalization of the Banzhaf Power Index using recent statistics.

Suppose we wish to evaluate the BPI of an undecided voter in Arizona, a fairly balanced state in terms of voting members that affiliate with either the Democratic or Republican Party. Using data from a 2016 Gallup Poll we see that approximately 41.7% of the state's population identifies as a Republican, and 41.4% as a Democrat. The poll estimates that approximately 16.9% of Arizona's population does not affiliate with either party. For the General Presidential Election in 2016, 3,588,466 people registered for voting, and were eligible to vote on November 8th. For purpose of practice, let's assume these numbers to be accurate and true. Also, let's make the assumption that all those who identify as a Republican or Democrat will definitely vote with that party on voting day, regardless of the candidate. Now, let's make use of our generalized formula for BPI of an undecided voter.

Let's state the necessary variables. We will use the estimation of the Republican affiliation for P_A, and Democratic affiliation for P_B:

$$P_A = \frac{(0.5 \cdot 3,588,466) - (.417 \cdot 3,588,466)}{(.169 \cdot 3,588,466)} \approx .4911$$

$$P_B = \frac{(0.5 \cdot 3,588,466) - (.414 \cdot 3,588,466)}{(.169 \cdot 3,588,466)} \approx .5089$$

$$u = .169 \cdot 3,588,466 \approx 606,450 \ .$$

Plugging in these results, we have:

$$P_c(u) \approx \left[(.4911)^{-.4911} \cdot (.5089)^{-.5089} \cdot \frac{1}{2}\right]^{606,450} \cdot \frac{1}{\sqrt{2\pi(606,450)(.4911)(.5089)}} \approx 1.923014088 \cdot 10^{(-45)}.$$

As shown, this is an extremely small probability; however, relative to voting power in other states, this result is not extreme. Let's take a look at power metrics in other states, and proportions of the population that identify as Democrats or Republicans:

State	% Rep.	% Dem.	$P_c(u)$
Arizona	41.7	41.4	$1.923 \cdot 10^{-45}$
Colorado	43.0	42.4	$7.291 \cdot 10^{-180}$
Indiana	45.2	37.2	$4.841 \cdot 10^{-39582}$

We find that as the decided voters "tilt" unevenly in one direction (within a state), the probability that an undecided voter swings the election decreases dramatically. Penrose mentions this in his paper:

"... a small 'resolute' group of people, who always vote together can exercise a surprisingly powerful control over the whole committee... a bloc of size $\sqrt{n}$ always can carry 84 percent of the decision."

To summarize, Penrose states that even a very small proportion of the population ($\sqrt{n}$) can have a very strong influence on the overall decision of the group (at least 84%), if they decide to vote together.

With these ideas in mind, our focus will shift. Now, we can look at the difference in predetermined voters. What we see is that opposing preallocated votes tend to balance the scales in terms of voting power. The remaining block of voters on either side will have this "surprisingly powerful control" that Penrose mentions. Let's take a look at a visual of the swing state's voting population:

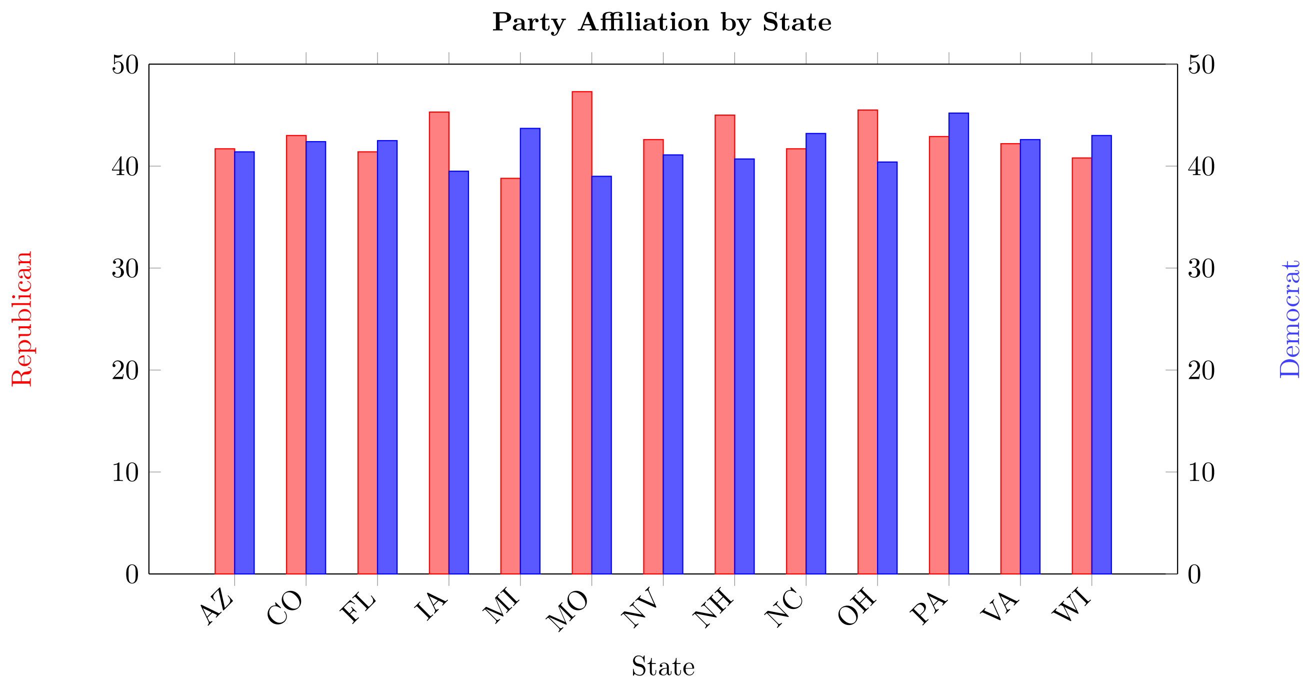


Figure 1: Based on data from 2016 Gallup Poll

When observing this graph, notice the *difference* between the proportion of Democrats and Republicans within each state. The following table will focus on this statistic:

State	(% Rep) - (% Dem)	$P_c(u)$
Arizona	0.30	$1.923 \cdot 10^{-45}$
Colorado	0.60	$7.291 \cdot 10^{-180}$
Florida	-1.10	$1.068 \cdot 10^{-2109}$
Iowa	5.80	$3.578 \cdot 10^{-10708}$
Michigan	-4.90	$1.284 \cdot 10^{-22512}$
Missouri	8.30	$6.376 \cdot 10^{-49459}$
Nevada	1.50	$2.493 \cdot 10^{-507}$
New Hampshire	4.30	$3.253 \cdot 10^{-2874}$
North Carolina	-1.50	$8.315 \cdot 10^{-2249}$
Ohio	5.10	$1.206 \cdot 10^{-32235}$
Pennsylvania	-2.30	$1.566 \cdot 10^{-8470}$
Virginia	-0.40	$4.545 \cdot 10^{-133}$
Wisconsin	-2.20	$1.259 \cdot 10^{-2319}$

Additionally, these results depend upon the number of registered voters within the state, but this table gives us a good idea of the bigger picture. Note that these figures are simply describing an individual's voting power within a state. Recall Penrose's assertion about a majority bloc's overwhelming ability to dominate voting situations. Let's take a look at how these bloc voters in the United States could affect their state's undecided voters in this theoretical situation. This table displays undecided voter BPIs for different levels of preallocation. If 0% of voters are assumed to be predetermined, we should obtain Penrose's square root that we discussed earlier. If 20% of each polled partisan voter was assumed to be predetermined, we should obtain a significantly lower BPI for an undecided voter, as was discussed earlier. First, California:

% Partisan	$P_c(u)$	P(Bloc Win)
0	0.000179	0.50
20	$2.042 \cdot 10^{-6769}$	1.00
50	$4.602 \cdot 10^{-60218}$	1.00
70	$3.088 \cdot 10^{-165943}$	1.00
100	0.0	1.00

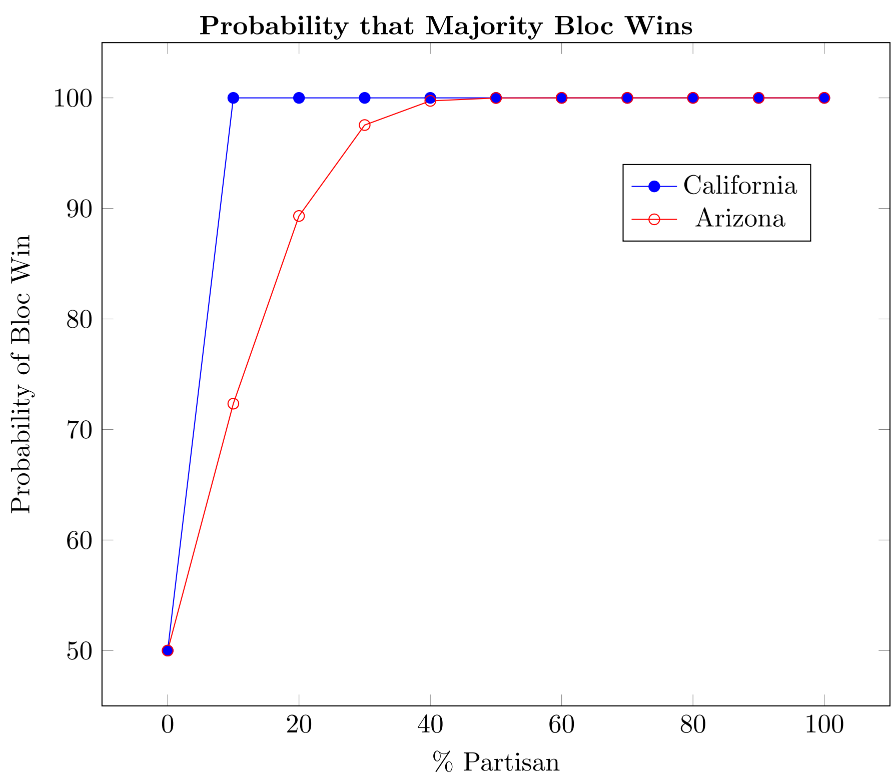
If a relatively small proportion of predetermined voters are preallocated, an undecided voter in California holds nearly zero chance of being influential in an election.

A voter in Arizona, however, might have a more optimistic story to tell. Arizona, in recent years, has been considered a swing state (i.e., the difference in number in Republican vs. Democratic voters is notably smaller).

% Partisan	$P_c(u)$	P(Bloc Win)
0	0.00042	0.50
20	0.000177	0.893
50	$4.594 \cdot 10^{-7}$	0.999
70	$4.623 \cdot 10^{-12}$	1.00
100	$1.923 \cdot 10^{-45}$	1.00

In a more balanced state, such as Arizona, undecided voters have a significantly greater potential for influence, as we assume higher proportions of partisan voters are in fact predetermined.

The graph below illustrates how states with voting blocs of greater magnitude almost surely take away the majority of influence of an undecided voter.



What if we wanted to calculate the individual's voting power at the national level? Here, we may simply multiply the individual's voting power within his/her state by the state's voting power within the overall voting system (in this case, the electoral college) due to the independent nature of these two probabilities. To calculate a swing state's voting power in the electoral college, we made assumptions about 38 out of the 51 voting "states" within the electoral college, and calculated BPIs for the swing states based off these assumptions using a programming language. Simply, we generated scenarios in which these swing states would be critical given that the other 38 states are "predetermined" voters. We decided which way these 38 states would vote (either republican or democrat) based on a brief history of their partisan tendencies. California, for example, we assume will vote for the Democratic Party, while we assume Texas will vote for the Republican Party.

Presented below is a table that gives the individual voting power at the national level for swing states, assuming the numbers in the Gallup poll are true and accurate. Take note of the following notation:

$$\text{Undecided Voter BPI Within State} \rightarrow BPI_I(u)$$

$$\text{States BPI Within Electoral System} \rightarrow BPI_S(S)$$

$$\text{Undecided Voter BPI at National Level} \rightarrow BPI_N(u)$$

BPIs are extremely small at the national level for individual, undecided voters due to its multiplicative nature:

$$BPI_N(u) = P_c(u) \cdot BPI_S(S) \ . \tag{8}$$

State	$BPI(u)$	$BPI_S(S)$	$BPI_N(u)$
Arizona	$1.923 \cdot 10^{-45}$	0.0653	$1.255719 \cdot 10^{(-46)}$
Colorado	$7.291 \cdot 10^{-180}$	0.0526	$3.835066 \cdot 10^{(-181)}$
Florida	$1.068 \cdot 10^{-2109}$	0.1832	$1.956576 \cdot 10^{(-2110)}$
Iowa	$3.578 \cdot 10^{-10708}$	0.0355	$1.27019 \cdot 10^{(-10709)}$
Michigan	$1.284 \cdot 10^{-22512}$	0.0958	$1.230072 \cdot 10^{(-22513)}$
Missouri	$6.376 \cdot 10^{-49459}$	0.0588	$3.749088 \cdot 10^{(-49460)}$
Nevada	$2.493 \cdot 10^{-507}$	0.0355	$8.85015 \cdot 10^{(-509)}$
New Hampshire	$3.253 \cdot 10^{-2874}$	0.0225	$7.31925 \cdot 10^{(-2876)}$
North Carolina	$8.315 \cdot 10^{-2249}$	0.0891	$7.408665 \cdot 10^{(-2250)}$
Ohio	$1.206 \cdot 10^{-32235}$	0.1067	$1.2868 \cdot 10^{(-32236)}$
Pennsylvania	$1.566 \cdot 10^{-8470}$	0.1205	$1.887 \cdot 10^{(-8471)}$
Virginia	$4.545 \cdot 10^{-133}$	0.0753	$3.422385 \cdot 10^{(-134)}$
Wisconsin	$1.259 \cdot 10^{-2319}$	0.0588	$7.4029 \cdot 10^{(-2321)}$

Obtaining probabilistic measures for undecided voters within their own unweighted voting system (the state the voter registers with) coupled with the BPI for the corresponding state within the state's weighted voting system (the Electoral College), we can shed light onto the intricacies of partisanship in U.S. voting.

Recap

After reading through the mathematical proofs, tables, and data provided throughout this article, one might become skeptical of their purpose as a voter. Academic research that sparks doubt, conflict, or provocation often leads to positive change. It must be reinforced, however, that exercising voting rights is of monumental importance in the inherent principles of a democracy.

The notion of voting power gives us meaningful and significant mathematical insight into the response to a few questions posed towards the beginning of this paper. To be clear, the purpose of this paper is not suggest or discredit any certain type of voting system, but simply to further demonstrate the potential limitations of weighted voting systems.

Acknowledgements

The authors would like to thank Southern Nazarene University, along with the NASA Oklahoma Space Grant Consortium for their monumental contributions to this project.

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