NTH DIFFERENCE OF THE NTH POWER

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Elementary Pattern Recognition

Sample Pattern 1:	0	1	3	6	10	15	…
	+1	+2	+3	+4	+5	+?	…

Sample Pattern 2:	1	2	4	8	16	32	…
	x2	x2	x2	x2	x2	x2	
	+1	+2	+4	+8	+16	+?	…

Sample Pattern 3:	1	4	9	16	25	36	…
	1^2	2^2	3^2	4^2	5^2	6^2	…

DIFFERENCE OF SQUARES

Perfect Squares:

0	1	4	9	16	25	36	49	…
0^2	1^2	2^2	3^2	4^2	5^2	6^2	7^2	…
+1	+3	+5	+7	+9	+11	+13	+15?	…

+2 +2 +2 +2 +2 +2 +2 …

DIFFERENCE OF CUBES??

Perfect Cubes:

0^3	1^3	2^3	3^3	4^3	5^3	6^3	7^3	…
0	1	8	27	64	125	216	343	…
	+1	+7	+19	+37	+61	+91	+127	…
		+6	+12	+18	+24	+30	+36	…
			+6	+6	+6	+6	+6	…

Difference of the nth Power 1-3

X^1:

0^1		1^1		2^1		3^1		4^1		5^1		6^1		7^1		8^1
0		1		2		3		4		5		6		7		8
	1		1		1		1		1		1		1		1	

X^2:

0^2		1^2		2^2		3^2		4^2		5^2		6^2		7^2		8^2
0		1		4		9		16		25		36		49		64
	1		3		5		7		9		11		13		15	
		2		2		2		2		2		2		2		

X^3:

0^3		1^3		2^3		3^3		4^3		5^3		6^3		7^3		8^3
0		1		8		27		64		125		216		343		512
	1		7		19		37		61		91		127		169	
		6		12		18		24		30		36		42		
			6		6		6		6		6		6			

Difference of the Nth Power 4-5

X^4:

0^4	1^4	2^4	3^4	4^4	5^4	6^4	7^4	8^4
0	1	16	81	256	625	1296	2401	4096
1	15	65	175	369	671	1105	1695	
14	50	110	194	302	434	590		
36	60	84	118	132	156			
24	24	24	24	24				

X^5:

0^5	1^5	2^5	3^5	4^5	5^5	6^5	7^5	8^5
0	1	32	243	1024	3125	7776	16807	32768
1	31	211	781	2101	4651	9031	15961	
30	180	570	1320	2550	4380	6930		
150	390	750	1230	1830	2550			
240	360	480	600	720				
120	120	120	120					

NTH DIFFERENCE OF THE NTH POWER: PROPOSAL #1

For any given integer n ≥ 1, the n^{th} difference of any consecutive set of (n+1) integers raised to the n^{th} power is equal to n!

X⁴:

0^4	1^4	2^4	3^4	4^4	5^4	6^4	7^4	8^4
0	1	16	81	256	625	1296	2401	4096

1 15 65 175 369 671 1105 1695

14 50 110 194 302 434 590

36 60 84 118 132 156

24 24 24 24 24

MODELING THE NTH DIFFERENCE

X^2:

0^2		**1^2**		**2^2**		**3^2**		**4^2**		**5^2**		**6^2**		**7^2**		**8^2**
0		1		4		9		16		25		36		49		64
	1		3		5		7		9		11		13		15	
		2		2		2		2		2		2		2		

2 = 5 – 3
= (9-4) – (4-1)
= 9 – 4 – 4 + 1
= $\mathbf{3^2 - 2 \cdot 2^2 + 1^2}$

2 = 15 – 13
= (64-49) – (49-36)
= 64 – 49 – 49 + 36
= $\mathbf{8^2 - 2 \cdot 7^2 + 6^2}$

2 = 11 – 9
= (36-25) – (25-16)
= 36 – 25 – 25 + 16
= $\mathbf{6^2 - 2 \cdot 5^2 + 4^2}$

Modeling the nth Difference

2 $= 11 - 9$
$= (36-25) - (25-16)$
$= 36 - 25 - 25 + 16$
$= \mathbf{6^2 - 2 \cdot 5^2 + 4^2}$

2 $= 15 - 13$
$= (64-49) - (49-36)$
$= 64 - 49 - 49 + 36$
$= \mathbf{8^2 - 2 \cdot 7^2 + 6^2}$

Regardless of the value of the consecutive base integers, which could be called b, b + 1, and b + 2, the value $(b+2)^2 - 2(b+1)^2 + b^2$ always equals 2

*2 $= 3 - 1$
$= (4-1) - (1-0)$
$= 4 - 1 - 1 + 0$
$= \mathbf{2^2 - 2 \cdot 1^2 + 0^2}$

*Base Zero Case

Note that the Base Zero case, when b=0, is especially easy to evaluate since its base integers are the integers 0 – *n*

NTH DIFFERENCE > 2

X^3:

0^3		1^3		2^3		3^3
0		1		8		27
	1		7		19	
		6		12		
			6			

n = 3; CD = 3! = 6

$6 = 12 - 6$

$= (19 - 7) - (7 - 1)$

$= [(27 - 8) - (8 - 1)] - [(8 - 1) - (1 - 0)]$

$= [(3^3 - 2^3) - (2^3 - 1^3)] - [(2^3 - 1^3) - (1^3 - 0^3)]$

$= 3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 - 0^3$

X^4:

0^4		1^4		2^4		3^4		4^4
0		1		16		81		256
	1		15		65		175	
		14		50		110		
			36		60			
				24				

n = 4; CD = 4! = 24

$24 = 60 - 30$

$= (110 - 50) - (50 - 14)$

$= [(175 - 65) - (65 - 15)] - [(65 - 15) - (15 - 0)]$

$\{ [(4^4 - 3^4) - (3^4 - 2^4)] - [(3^4 - 2^4) - (2^4 - 1^4)]\} - \{[(3^4 - 2^4) - (2^4 - 1^4)] - [(2^4 - 1^4) - (1^4 - 0^4)]\}$

$= 4^4 - 4 \cdot 3^4 + 6 \cdot 2^4 - 4 \cdot 1^4 + 0^4$

Nth Difference Expansions

n=1: Common Difference = n! = 1! = 1 = $1^1 - 0^1$

n=2: Common Difference = n! = 2! = 2 = $2^2 - 2 \cdot 1^2 + 0^2$

n=3: Common Difference = n! = 3! = 6 = $3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 - 0^3$

n=4: Common Difference = n! = 4! = 24 = $4^4 - 4 \cdot 3^4 + 6 \cdot 2^4 - 4 \cdot 1^4 + 0^4$

n=5: Common Difference = n! = 5! = 120 = $5^5 - 5 \cdot 4^5 + 10 \cdot 3^5 - 10 \cdot 2^5 + 5 \cdot 1^5 - 0^5$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

n=1 $1 \cdot 1^1 - 1 \cdot 0^1$

n=2 $1 \cdot 2^2 - 2 \cdot 1^2 + 1 \cdot 0^2$

n=3 $1 \cdot 3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 - 1 \cdot 0^3$

n=4 $1 \cdot 4^4 - 4 \cdot 3^4 + 6 \cdot 2^4 - 4 \cdot 1^4 + 1 \cdot 0^4$

n=5 $1 \cdot 5^5 - 5 \cdot 4^5 + 10 \cdot 3^5 - 10 \cdot 2^5 + 5 \cdot 1^5 - 1 \cdot 0^5$

Coefficients reflect Pascal's Triangle

Modeling Pascal

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{n}{t} = \frac{n!}{t!(n-t)!}$$

NTH DIFFERENCE MODEL

n=1: Common Difference = n! = 1! = 1 = $1^1 - 0^1$

n=2: Common Difference = n! = 2! = 2 = $2^2 - 2\cdot1^2 + 0^2$

n=3: Common Difference = n! = 3! = 6 = $3^3 - 3\cdot2^3 + 3\cdot1^3 - 0^3$

n=4: Common Difference = n! = 4! = 24 = $4^4 - 4\cdot3^4 + 6\cdot2^4 - 4\cdot1^4 + 0^4$

n=5: Common Difference = n! = 5! = 120 = $5^5 - 5\cdot4^5 + 10\cdot3^5 - 10\cdot2^5 + 5\cdot1^5 - 0^5$

$$\sum_{t=0}^{n} t^n \, (-1 \sum_{t=b}^{b+n} t^n \binom{n}{n+b-t} (-1)^{n+b+t}$$

NTH DIFFERENCE OF THE NTH POWER: REVISED PROPOSAL

For any given integer n ≥ 1, the n^th^ difference of any consecutive set of (n+1) integers raised to the n^th^ power is equal to n!

$$\sum_{t=0}^{n} t^{n}(-1)^{n+t}\binom{n}{t} = n!$$

Our Answer Was Stirling

Recall that a Stirling Number of the second kind is the number of ways to partition a set of n objects into k non-empty subsets and is denoted by $S(n,k)$ or $\left\{ {n \atop k} \right\}$.

Definition. $\left\{ {n \atop k} \right\} = \sum_{j=1}^{k} (-1)^{k-j} \dfrac{j^{n-1}}{(j-1)!(k-j)!}$

$= \sum_{j=1}^{k} (-1)^{k-j} \dfrac{j^{n}}{j!(k-j)!}$

$= \dfrac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^{n}$

Lemma 1. $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$, since there is only one way to partition n numbers into either 1 subset or n subsets.

Proof.

$\left\{ {n \atop 1} \right\} = \sum_{j=1}^{1} (-1)^{1-j} \dfrac{j^{n-1}}{(j-1)!(1-j)!} = (-1)^{0} \dfrac{(1)^{n-1}}{(0)!(0)!} = 1^{n-1} = 1$

$\left\{ {n \atop n} \right\} = \sum_{j=1}^{n} (-1)^{n-j} \dfrac{j^{n-1}}{(j-1)!(n-j)!} = \dfrac{1}{n!} \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} j^{n} = \dfrac{1}{n!}(n!) = 1$

Stirling Extrapolated

Part 1: Base Zero Case.
$\forall N \in \mathbb{Z}^+ \mid \; N \neq 0, \; \sum_{t=0}^{N} (t^N)(-1)^{N+t}\binom{N}{t} = N!$

Proof.

$$\begin{aligned}
\sum_{t=0}^{N} (t^N)(-1)^{N+t}\binom{N}{t} &= (-1)^N \sum_{t=0}^{N} (t^N)(-1)^{t}\binom{N}{t} \text{ (by factoring)} \\
&= (-1)^N((-1)^N \left\{ {N \atop N} \right\} N!) \text{ (by Lemma 2)} \\
&= (-1)^{2N}(N!) \text{ (by Lemma 1 and 2)} \\
&= N!
\end{aligned}$$

Case Closed?!

For any given integer n ≥ 1, the n^{th} difference of any consecutive set of (n+1) integers raised to the n^{th} power is equal to
n!

X^4:

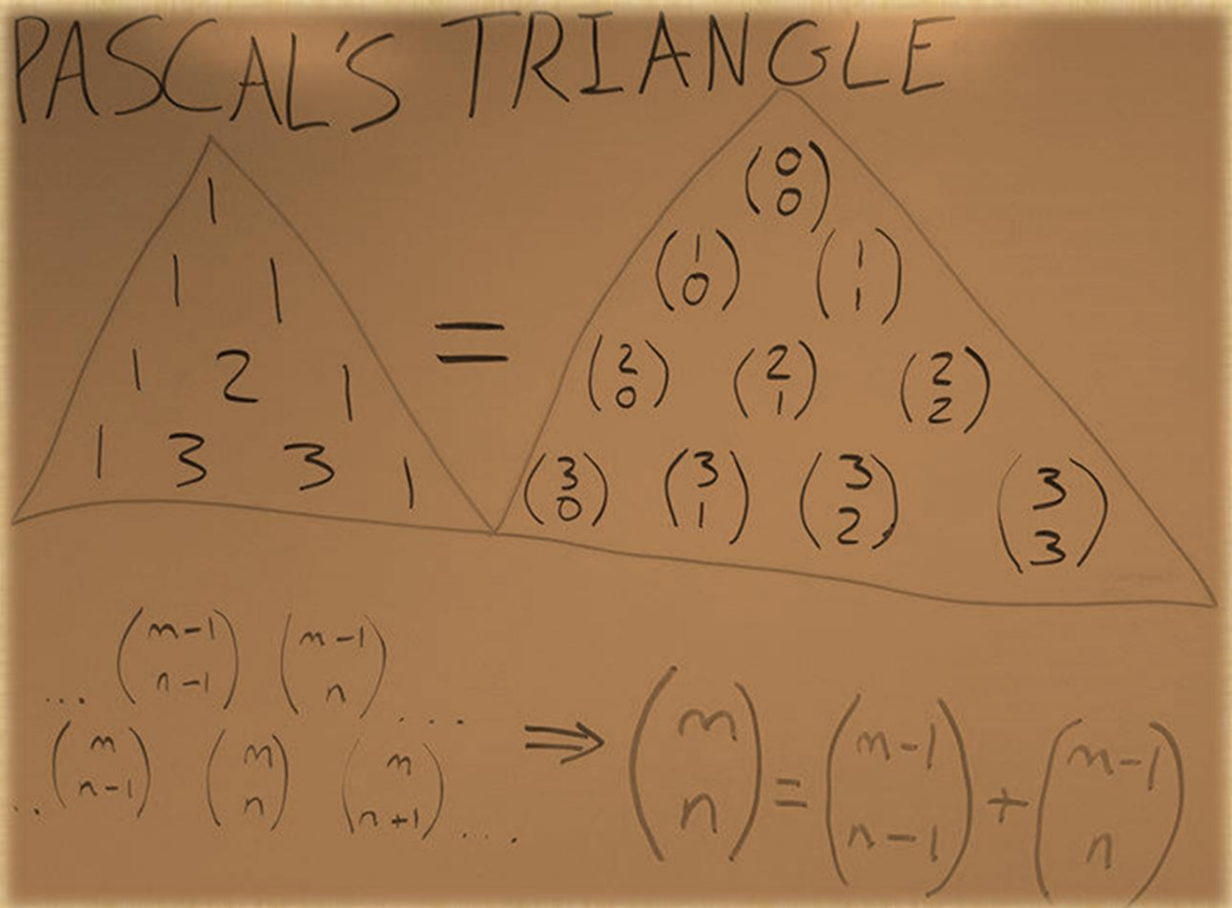
0^4	1^4	2^4	3^4	4^4	5^4	6^4	7^4	8^4
0	1	16	81	256	625	1296	2401	4096
1	15	65	175	369	671	1105	1695	
14	50	110	194	302	434	590		
36	60	84	118	132	156			
24	24	24	24	24				

Extending the Case

$$\sum_{t=b}^{N+b} (t^N)(-1)^{(N+b)+t} \binom{N}{t-b}$$

Nth Difference from any base of consecutives

Pascal's Combinatorics



Terms of Difference Triangles

DIFFERENCE TRIANGLES

$D(0,0)$ $D(0,1)$ $D(0,2)$ $D(0,3)$

$D(1,0)$ $D(1,1)$ $D(1,2)$

$D(2,0)$ $D(2,1)$

$D(3,0)$

$\ddots$

$D(0,n) \in \mathbb{R}$

$$D(m,n) = D(m-1,n+1) - D(m-1,n)$$

EXTENDING THE CASE

GIVEN:

$$D(m,n) = D(m-1,n+1) - D(m-1,n)$$

$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}$$

REPRESENTATION OF (m,n)th DIFFERENCE

Ex |

$$D(2,d-1) = \binom{1}{0} D(1,d) - \binom{1}{1} D(1,d-1)$$

$$= \left[\binom{1}{0} D(0,d) - \binom{1}{1} D(0,d-1)\right]$$

$$- \left[\binom{1}{0} D(0,d-1) - \binom{1}{1} D(0,d-2)\right]$$

$$D(2,d-1) = \binom{2}{0} D(0,d) - \binom{2}{1} D(0,d-1) + \binom{2}{2} D(0,d-2)$$

Extending the Case

GIVEN:

$$D(m,n) = D(m-1, n+1) - D(m-1, n)$$

$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}$$

REPRESENTATION OF (m,n)th DIFFERENCE

Ex

$$\begin{aligned} & D(2, d-1) \\ - & D(2, d-2) \end{aligned} = \begin{aligned} & \left[\binom{2}{0}D(0,d+1) - \binom{2}{1}D(0,d) + \binom{2}{2}D(0,d-1)\right] \\ - & \left[\binom{2}{0}D(0,d) - \binom{2}{1}D(0,d-1) + \binom{2}{2}D(0,d-2)\right] \end{aligned}$$

$$D(3, d-2) = \binom{3}{0}D(0, d+1) - \binom{3}{1}D(0,d) + \binom{3}{2}D(0,d-1) - \binom{3}{3}D(0,d-2)$$

$\Rightarrow$ Difference $$D(m,n) = \sum_{k=0}^{m} \binom{m}{k}(-1)^k D(0, m+n-k)$$

Extending the Case

REPRESENTATION OF THE (m,n)th DIFFERENCE

$$D(m,n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k D(0, m+n-k)$$

Ex | Series k^p ; $p=3$

m \ n	0	1	2	3	4	5 ...
0	0	1	8	27	64	125 ...
1	1	7	19	37	61 ...	
2	6	12	18	24 ...		
3	6	6	6 ...			
4	0	0 ...				
⋮	⋮					

Extending the Case

REPRESENTATION OF THE (m,n)th DIFFERENCE

$$D(m,n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k D(0, m+n-k)$$

If $D(0,d) = d^p$; $p = 1, 2, \ldots$

Then $$D(m,n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m+n-k)^p$$

$$D(m,0) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m-k)^p$$

$$^{*}D(0, m+n-k) = (m+n-k)^p$$

when $D(0,k) = k^p$

Extending the Case

REPRESENTATION OF THE (m,n)th DIFFERENCE

$$D(m,n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k D(0, m+n-k)$$

Given $D(0,d) = d^p$; $n, p \in \mathbb{Z}^+ \ni n, p = 1, 2, \ldots$

$$D(m,0) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m-k)^p$$

If m is EVEN, $(m+1) \bmod 2 \Rightarrow 1$

If m is ODD, $(m+1) \bmod 2 \Rightarrow 0$

$$D(m,0) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{k+(m+1) \bmod 2} k^p$$

Extending the Case

REPRESENTATION OF THE (m,n)th DIFFERENCE

$$D(m,n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k D(0, m+n-k)$$

Given $D(0,d) = d^p$; $n,p \in \mathbb{Z}^+ \ni n,p = 1,2,\ldots$

$$D(m,0) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{k+(m+1)\bmod 2} k^p$$

☆ If $m=p$, as in n^{th} difference research

$$D(p,0) = \sum_{k=0}^{p} \binom{p}{k} (-1)^{k+(p+1)\bmod 2} k^p$$

$$= p!\, S_p^{(m=p)} = p!$$

Extending the Case

EVALUATING THE (p,n)th DIFFERENCE

$$D(m,n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k D(0, m+n-k)$$

Given $D(0,d) = d^p$; $n,p \in \mathbb{Z}^+ \ni n,p = 1,2,\ldots$

$$D(p,0) = \sum_{k=0}^{p} \binom{p}{k} (-1)^{k+(p+1) \bmod 2} k^p = p!\, S_p^{(p)} = p!$$

$$D(p+1,0) = \sum_{k=0}^{p+1} \binom{p+1}{k} (-1)^{k+p \bmod 2} k^{p+1} = (p+1)!\, S_p^{(p+1)} = 0$$

Evaluation of the Extension

EVALUATING THE (p,n)th DIFFERENCE

$$D(m,n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k D(0, m+n-k)$$

Given $D(0,d) = d^p$; $n,p \in \mathbb{Z}^+ \ni n,p = 1,2,\ldots$

$$D(p,0) = p!\, S_p^{(p)} = p! \quad ; \quad D(p+1,0) = (p+1)!\, S_p^{(p+1)} = 0$$

$$\Rightarrow D(p+1,0) = D(p,1) - D(p,0)$$

$$\Rightarrow \quad D(p,0) = D(p,1) = p!$$

$\Rightarrow$ SIMILARLY $\underline{D(p,0) = D(p,n) = p!}$

CONCLUSION

Our proposal

For any given integer n ≥ 1, the n^{th} difference of any consecutive set of (n+1) integers raised to the n^{th} power is equal to n!

has been shown to be true, since the (n+1)st difference, which must be the difference of two numbers on the (n)th difference level, will always be a zero given an original series of integers raised to the (n)th power.

Therefore, all numbers on the (n)th difference level must be equal to each other. Thus, the (n)th difference level is a common difference of n!

$$\sum_{t=b}^{N+b} (t^N)(-1)^{(N+b)+t} \binom{N}{t-b} = N!$$

□