Benefits of Early Rigor and a Forms Approach to Limits In Calculus 1

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Outline

General Argument: Epsilon-Delta Proofs

1. "Students don't *get* limits (and continuity)," so textbooks usually make haste to get to derivatives.
2. Limits and continuity *are* technically challenging.
3. However, so are other parts of calculus.
4. Later parts of calculus seem straightforward enough, after rigorous limits/continuity training, and especially ε-δ proofs!
   (a) Make students really wrap their brains around what happens at the "micro" level.
   (b) At first, nobody "gets" ε-δ.
   (c) With work, everyone can, especially for continuity. (Equalizer?)
   (d) After that, much more straightforward are: differentiation rules (e.g., Chain Rule), implicit differentiation, logarithmic differentiation, and eventually integration!

Purdue Summer Calculus 1990 versus 1993

The taught Calculus 1 (five hours) as the instructor (still a teaching assistant) these two summers. Impressions:

Summer 1990: Department included ε-δ computations for tolerances, proofs for limits (1.5 lessons).

- Students did not "get" proofs; unhappy.
- But most could be made to mimic the proofs.
- Instructor could still refer back to the idea when limit technicalities arose.
- Used Richard Hunt's book. Later problems solving inequalities arising from the Mean Value Theorem were fairly straightforward.

Summer 1993: No longer taught ε-δ. (Students "won.")

- Later limit arguments seemed more *ad hoc.*
- Students were strikingly less able to do the Mean Value Theorem applications.
  - Because they had less practice with inequalities?
  - Because they had less practice with clever—and complex—technical points?

Defining Continuity, then Limits, by Epsilon-Delta

I. $f(x)$ is continuous at $x = a$

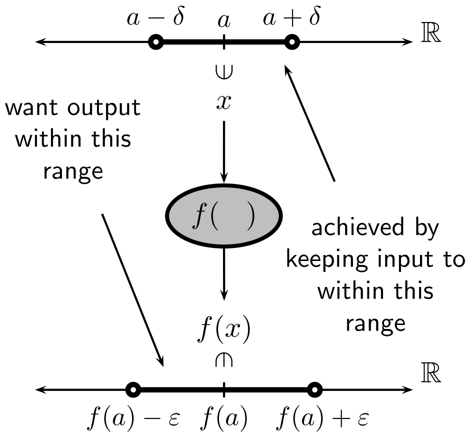
$$\iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon].$$

1. Requires some knowledge of quantifiers, absolute value inequalities. Warm up: "For every man there is a woman who loves him":

$$(\forall m \in M)(\exists w \in W)[w \text{ loves } m].$$

2. Move quantifiers around, learn how to negate these, truth tables, equivalent statements, negating statements, e.g., $\sim (P \to Q) \iff P \wedge (\sim Q)$.

3. (Shhh.....In fact, what makes it all work is teaching them symbolic logic first for a week.)


$a - \delta$
a
$a + \delta$
$\mathbb{R}$
$\Cup$
x
want output within this range
$f(\quad)$
achieved by keeping input to within this range
$f(x)$
$\Cap$
$\mathbb{R}$
$f(a) - \varepsilon$
$f(a)$
$f(a) + \varepsilon$

Show that $f(x) = 9 - 4x$ is continuous at $x = 2$.

Scratch-work: Here $a = 2$ and $f(a) = f(2) = 1$.

$$\begin{array}{rrcl} & |f(x) - 1| & < & \varepsilon \\ \iff & |9 - 4x - 1| & < & \varepsilon \\ \iff & |-4x + 8| & < & \varepsilon \\ \iff & |(-4)(x-2)| & < & \varepsilon \\ \iff & 4|x-2| & < & \varepsilon \\ \iff & |x-2| & < & \frac{1}{4}\varepsilon. \end{array}$$

Proof: For $\varepsilon > 0$, choose $\delta = \frac{1}{4}\varepsilon$. Then $\delta > 0$ (exists) and

$$|x-2| < \delta \implies |f(x) - 1| = |9 - 4x - 1| = |-4x + 8| = |(-4)(x-2)|$$
$$= 4|x-2| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon, \text{ q.e.d.}$$

Larson's Limit Definition

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

- Some books write:
  "so that $|f(x) - L| < \varepsilon$ *whenever* $0 < |x - c| < \delta$."

- Compare to: $\lim_{x \to a} f(x) = L$

$$\iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon].$$

More Quantified Statements for Students to Ponder

- $f(x)$ is right-continuous at $x = a \iff$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[x \in [a, a + \delta) \longrightarrow |f(x) - f(a)| < \varepsilon]$$

- $\lim_{x \to a} f(x) = L \iff$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon]$$

- $\lim_{x \to a^+} f(x) = L \iff$

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[x \in (a, a + \delta) \longrightarrow |f(x) - L| < \varepsilon]$$

- $\lim_{x \to \infty} f(x) = L \iff$

$$(\forall \varepsilon > 0)(\exists N)(\forall x)[x > N \longrightarrow |f(x) - L| < \varepsilon].$$

- $\lim_{x \to \infty} f(x) = \infty \iff$

$$(\forall M)(\exists N)(\forall x)[x > N \longrightarrow f(x) > M].$$

General Argument: Forms Approach to Limits

A. Lets them learn how trends (in inputs) imply other trends (in outputs).

B. Also allows them to dabble in interesting limit-form arithmetic for determinant forms ("infinity bottles of beer on the wall..."):

$$1/0^+ = \infty, \quad 1/\infty = 0^+, \quad \infty+\infty = \infty, \quad \sin\frac{1}{\infty} = \sin 0^+ = 0^+,$$

$$\infty + 2 = \infty, \quad -5 \cdot \infty = -\infty, \quad B/\infty = 0, \quad B \cdot 0 = 0,$$

as well as to ponder indeterminant forms:

$$0/0, \quad \infty/\infty, \quad \infty - \infty, \quad 0 \cdot \infty, \quad \frac{\sin 0}{0}.$$

C. Makes them more ready for Calculus 2:

(1) L'Hôpital's Rule, recognizing $0/0$ and ∞/∞ forms.

(2) Further forms for improper integrals: $\tan^{-1}\infty = (\pi/2)^-$, $\ln(\ln(\ln\infty)) = \ln(\ln\infty) = \ln\infty = \infty$.

Example: Compute $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

Solution: Note that $x = 9$ is outside of the domain of the function, but the actual domain is $x \in [0, 9) \cup (9, \infty)$ so we can certainly approach $x = 9$ from both directions. More casually, we can say that we can let x venture small distances to the left or right of $x = 9$ and the function will be defined. The usual technique for a problem such as this is to algebraically rewrite it by multiplying by $(\sqrt{x}+3)/(\sqrt{x}+3)$:

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}$$

$$\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9} \frac{1}{\sqrt{x}+3} \overset{\text{CONT}}{=\!=} \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

Example: If possible find $\lim_{x\to 5^-}\sqrt{x^2-25}$ and $\lim_{x\to 5^+}\sqrt{x^2-25}$.

Solution:

$$\lim_{x\to 5^-}\sqrt{\underbrace{x^2-25}_{<0}} \quad \text{does not exist,}$$

$$\lim_{x\to 5^+}\sqrt{\underbrace{x^2-25}_{>0}} = \sqrt{5^2-25} \overset{\text{(Rt.) CONT}}{=\!=\!=} 0.$$

Example: Compute, if possible, $\lim_{x\to 0}\frac{x^2}{|x|}$.

Solution:

$$\left.\begin{aligned}
\lim_{x\to 0^-}\frac{x^2}{|x|} &\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 0^-}\frac{x^2}{-x} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 0^-}(-x) \overset{\text{CONT}}{=\!=} 0,\\
\lim_{x\to 0^+}\frac{x^2}{|x|} &\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 0^+}\frac{x^2}{x} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 0^+}(x) \overset{\text{CONT}}{=\!=} 0.
\end{aligned}\right\} \quad \therefore \lim_{x\to 0}\frac{x^2}{|x|} = 0.$$

1. $\lim_{x\to 0^-} \frac{1}{x^{2/3}} = \lim_{x\to 0^-} \frac{1}{\sqrt[3]{x^2}} \xlongequal{1/0^+} \infty$,

2. $\lim_{x\to -4} \frac{x}{(x+4)^{4/3}} = \lim_{x\to -4} \frac{x}{[(x+4)^4]^{1/3}} \xlongequal{-4/0^+} -\infty$,

3. $\lim_{x\to 6} \frac{x-4}{(x-6)^2} \xlongequal{2/0^+} \infty$,

4. $\lim_{x\to 6} \frac{x-4}{x-6} \xlongequal{2/0^\pm}$ DNE,

5. $\lim_{x\to \frac{\pi}{2}^-} \ln(\cos x) \xlongequal{\ln 0^+} -\infty$.

6. $\lim_{x\to 0} x \sin\frac{1}{x} \xlongequal{0\cdot B} 0$.

The Sandwich Theorem argument for the last limit above can be summarized graphically as follows:

$$\text{As } x \to 0: \quad \underbrace{-|x|}_{\substack{\downarrow \\ 0}} \;\le\; x\sin\tfrac{1}{x} \;\le\; \underbrace{|x|}_{\substack{\downarrow \\ 0}}$$

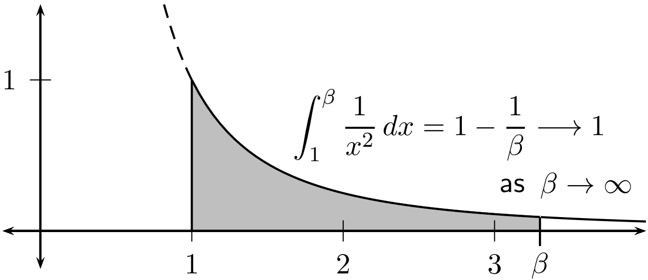
$$\therefore \;\; x\sin\tfrac{1}{x} \longrightarrow 0.$$

Alternatively, where B represents a bounded but well-defined quantity as $x \to 0$, we can write:

$$\lim_{x\to 0} x\sin\frac{1}{x} \overset{0\cdot B}{=\!=\!=} 0.$$

Similarly $\lim\limits_{x\to\infty}(x+\sin x) \overset{\infty + B}{=\!=\!=\!=} \infty.$

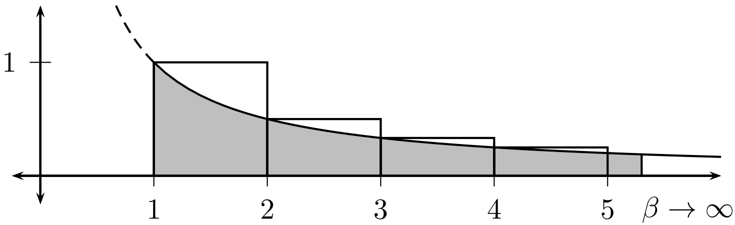
In Calculus 2, it is useful to have the notation
$x \to a \implies f(x) \to L$ as well as $\lim_{x \to a} f(x) = L$:



Alternatively,

$$\int_1^\infty \frac{1}{x^2}\,dx \overset{?}{=} \lim_{\beta\to\infty} \int_1^\beta \frac{1}{x^2}\,dx = \lim_{\beta\to\infty} \left[1 - \frac{1}{\beta}\right] \overset{1-\frac{1}{\infty}}{\underset{1-0}{=\!=\!=}} 1.$$

(Most don't object to using a "provisional =" in L'Hôpital's Rule problems, but many do with improper integrals, lest it be seen as a definition.)



For the series $\sum_{k=1}^{\infty} \frac{1}{k}$, we have partial sums

$$S_n > \int_1^{n+1} \frac{1}{x} = \ln(n+1) \xrightarrow[(\ln \infty)]{\ln(\infty+1)} \infty \text{ as } n \to \infty.$$

$\therefore \sum_{k=1}^{\infty} \frac{1}{k}$ diverges to ∞.

Conclusions (?)

A. Students can *get*—and do like—making limit forms arguments.

   1. The author suspects it helps students to better understand limiting trends in complex expressions.
   2. It makes many computations more elegant, and less awkward, at all levels of calculus.
   3. One must be sure "forms" and other arithmetic are separate:

      Correct: $\lim_{x\to\infty}(x+1) \overset{\infty+1}{=\!=\!=} \infty.$

      Questionable: $\lim_{x\to\infty}(x+1) = \infty + 1 = \infty.$

B. Students can also *get* ε-δ proofs, after which:

   1. other definitions using quantifiers can be offered and absorbed;
   2. students are so happy to see something as straightforward as, e.g., a quotient rule; inversely, if limits are given short shrift, perhaps it's okay to not quite understand derivatives and integrals?

C. So perhaps we should spend more time front-loading Calculus 1 with more work before derivatives? Shhh...helped with symbolic logic?