A Topological Definition of Limits for Use in Elementary Calculus

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Recall the definition for $\lim_{x \to c} f(x) = L$ from Calculus 1.

$\lim_{x \to c} f(x) = L$ iff for all $\varepsilon > 0$, there is a positive real number $\delta > 0$ such that for all x for which $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$.

While this definition has been the bane of all elementary calculus students since its introduction in the mid-19th century, it must be pointed out that it exonerated the calculus at a time it was being challenged by mathematicians and physicists alike.

The issue was that Newton and Leibnitz had never formally defined the idea of limit, they merely used it intuitively.

19th century mathematicians pointed out that without an explicit definition of limit, the calculus and all derived from it (physics) was suspect as being potentially logically inconsistent.

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The definition of limits saved the calculus as a valid mathematical entity but its use in verifying limit statements (both specific limits and theorems giving properties of limits) is at best tedious.

As far as specific limits, most elementary calculus texts settle for verifying limits of linear functions.

This is because that students matriculating into calculus have not developed the algebraic and geometric skills to adequately work with the δ-ε definition with functions other than linear functions.

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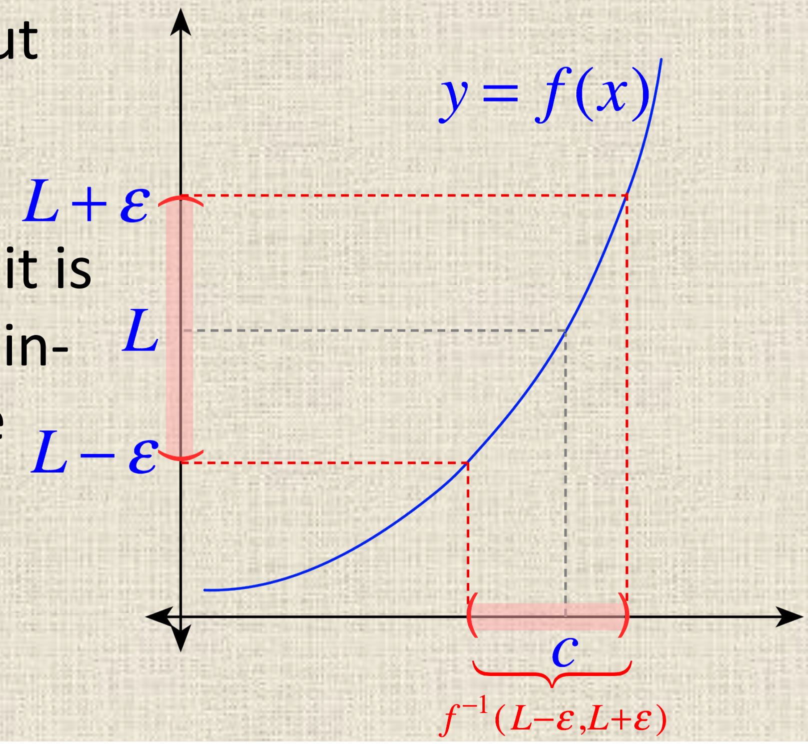
The primary difficulty lies in the fact that the definition geometrically demands that the open intervals as generated by the absolute value measures of the distances from c in the domain and L in the range must be centered about both c and L.

The usual procedure in verifying a limit is to analyze the inverse image of the ε-interval about the supposed limit in the range.



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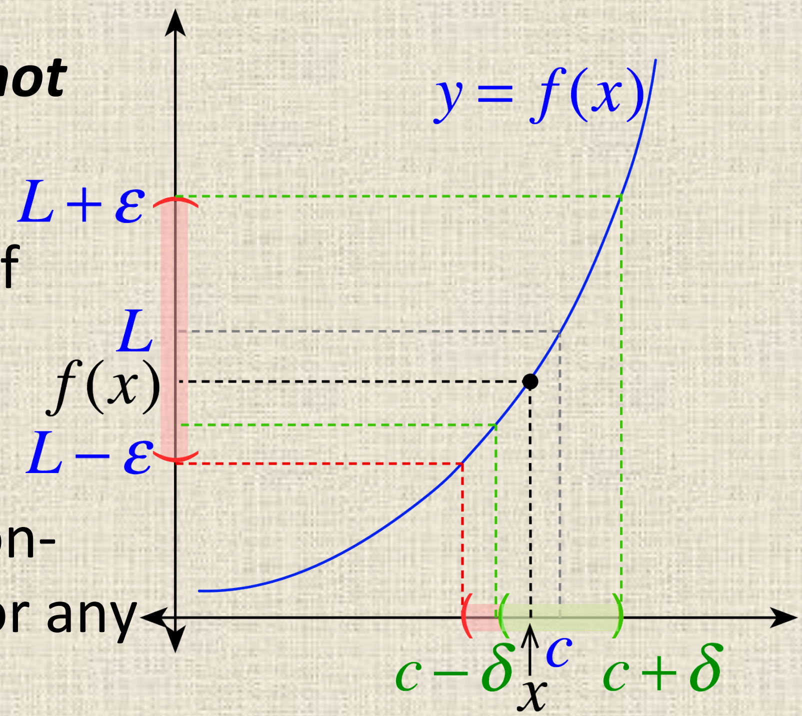
This inverse image will be an open set in the domain containing the real number c and thus this inverse image will contain an open interval about $x = c$.

In the vast majority of cases this will ***not*** be an interval *centered* at c.

Next we analyze how far either side of this interval extends and we let δ be the smaller of these 2 distances.

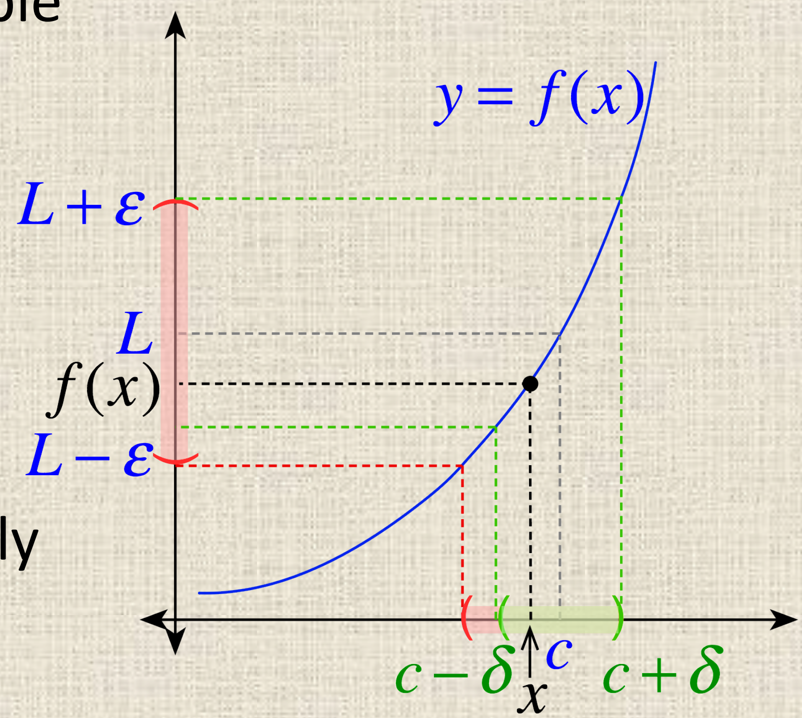
Clearly the δ-interval about $x = c$ is contained in $f^{-1}(L - \varepsilon, L + \varepsilon)$ and thus for any $x \in (c - \delta, c + \delta)$, $f(x) \in (L - \varepsilon, L + \varepsilon)$.



The problem students have is with determining the δ's, i.e. which of the two distances either side of c is smaller, which even in simple cases can be very difficult and involve a non-intuitive algebraic "trick" to do so.

Then the work to "show that the chosen δ works", i.e., given x in δ-interval about c that $f(x)$ is indeed in the ε-interval about L is generally even trickier – even for simple cases such as verifying $\lim_{x\to 2} x^2 = 4$.

Its been my experience over 35 years of teaching calculus that students do understand the idea behind the definition but they are unable to articulate the tricks involved in actually using it*.



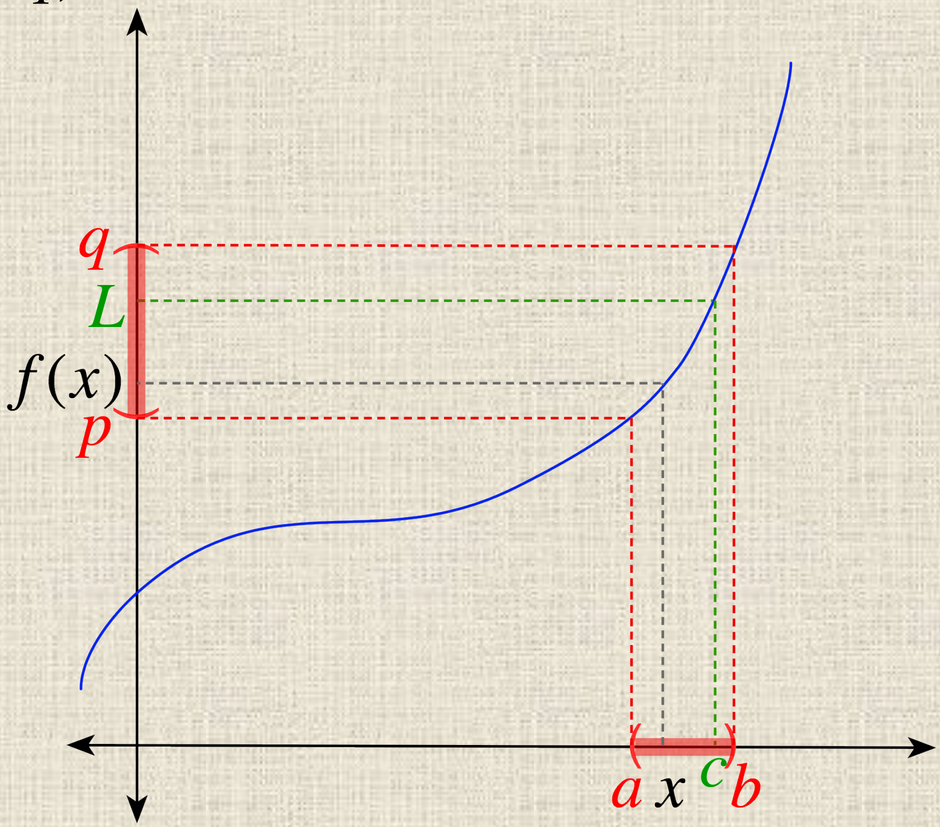
From a topological standpoint the δ-ε definition is way too strict and this is part of the reason for the difficulties in articulating it.

The bottom line is that the δ-ε definition is a means of quantifying what is meant by "closeness" and it intuitively can be paraphrased by: " $\lim_{x \to c} f(x) = L$ if and only if values of x close to c give values of $f(x)$ close to L".

From the standpoint of the topology of the real line, closeness to points is gaged by an open interval about that point, irrespective of whether it is centered or not.

It turns out that by replacing the centered intervals about the relevant points in the domain and range, as required by the δ-ε definition, by arbitrary open intervals many of the algebraic difficulties of articulating the limit definition disappear.

Definition (Topological). $\lim_{x \to c} f(x) = L$ if and only if for all open intervals (p, q) that contain L, there is an open interval (a, b) about $x = c$ in the domain such that if $x \in (a, b)$, $x \neq c$, then $f(x) \in (p, q)$.



An example of its use will show the utility of this definition.

Example 1. Show that $\lim_{x \to 2} x^3 = 8$.

Let (p, q) be an interval about 8 on the Y-axis.

Then $p < 8 < q$ and since the cube root function preserves inequalities: $\sqrt[3]{p} < 2 < \sqrt[3]{q}$.

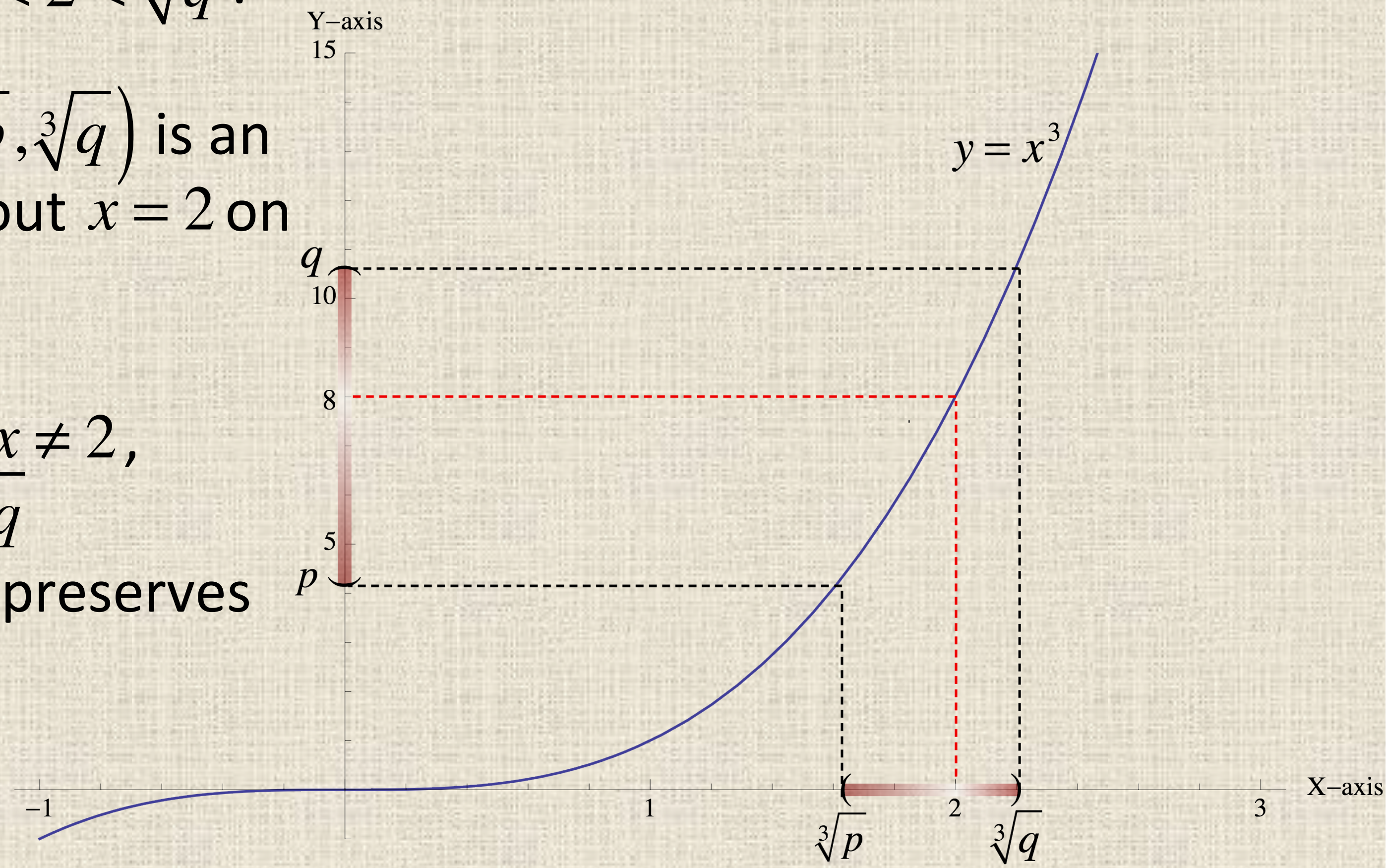
Thus $(a,b) = \left(\sqrt[3]{p}, \sqrt[3]{q}\right)$ is an open interval about $x = 2$ on the X-axis.

Given $x \in (a,b)$, $x \neq 2$,

$$\sqrt[3]{p} < x < \sqrt[3]{q}$$

and since cubing preserves inequalities:

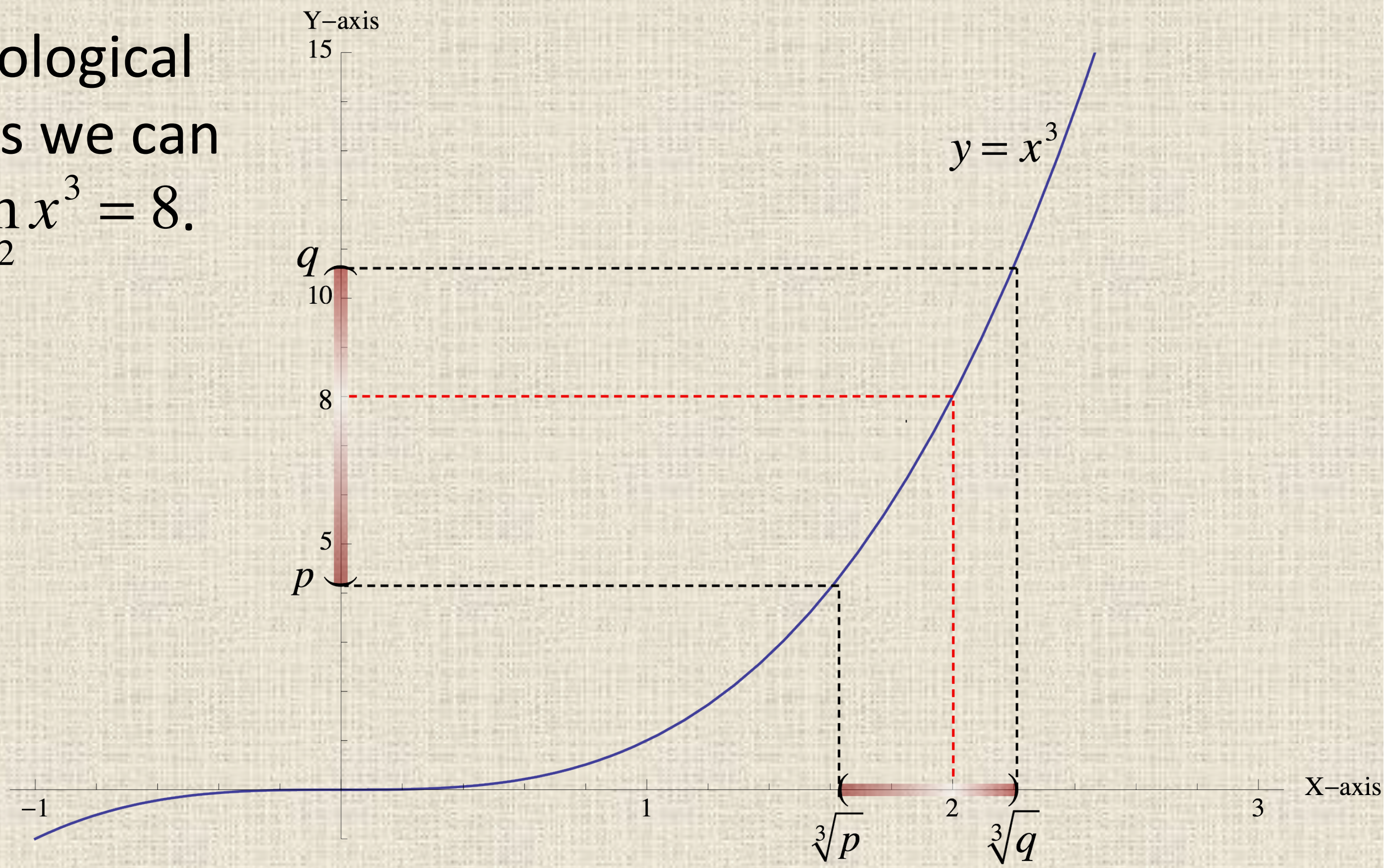
$$p < x^3 < q$$



Example 1. Show that $\lim_{x\to 2} x^3 = 8.$

Thus given an arbitrary open interval (p,q) about 8 on the Y-axis, there is always an interval about $x = 2$ on the X-axis such that if $x \in (a,b),\ x \neq 2$ then $f(x) = x^3 \in (p,q).$

Hence by the topological definition of limits we can conclude that $\lim_{x\to 2} x^3 = 8.$



Theorem. The Constant Multiple Property. Suppose $\lim_{x \to c} f(x) = L$ and that $\alpha \in \mathbb{R}$. Then $\lim_{x \to c} \alpha \cdot f(x) = \alpha \cdot L$.

Proof. Suppose that (p, q) is an open interval about $\alpha \cdot L$.

Case 1: $\alpha > 0$.

Since $p < \alpha \cdot L < q$ and $\alpha > 0$, $\frac{p}{\alpha} < \frac{\alpha \cdot L}{\alpha} < \frac{q}{\alpha} \Rightarrow \frac{p}{\alpha} < L < \frac{q}{\alpha}$.

Thus $(\frac{p}{\alpha}, \frac{q}{\alpha})$ is an open interval about L and since $\lim_{x \to c} f(x) = L$, the topological definition for limits implies there is an open interval, (a, b), about $x = c$ so that if $x \in (a, b)$, $x \neq c$, then $f(x) \in (\frac{p}{\alpha}, \frac{q}{\alpha})$.

Therefore for $x \in (a, b)$, $x \neq c$: $\quad \frac{p}{\alpha} < f(x) < \frac{q}{\alpha}$

$$\Rightarrow \alpha \cdot \left(\frac{p}{\alpha}\right) < \alpha \cdot f(x) < \alpha \cdot \left(\frac{q}{\alpha}\right)$$

So that $\alpha \cdot f(x) \in (p, q)$.

$$\Rightarrow p < \alpha \cdot f(x) < q$$

Theorem. The Constant Multiple Property. Suppose $\lim_{x\to c} f(x) = L$ and that $\alpha \in \mathbb{R}$. Then $\lim_{x\to c} \alpha \cdot f(x) = \alpha \cdot L$.

Proof. Suppose that (p,q) is an open interval about $\alpha \cdot L$.

Case 2: $\alpha < 0$.

Since $p < \alpha \cdot L < q$ and $\alpha < 0$, $\frac{p}{\alpha} > \frac{\alpha \cdot L}{\alpha} > \frac{q}{\alpha} \Rightarrow \frac{p}{\alpha} > L > \frac{q}{\alpha} \Rightarrow \frac{q}{\alpha} < L < \frac{p}{\alpha}$.

Thus $(\frac{q}{\alpha}, \frac{p}{\alpha})$ is an open interval about L and since $\lim_{x\to c} f(x) = L$, the topological definition for limits implies there is an open interval, (a,b), about $x = c$ so that if $x \in (a,b)$, $x \neq c$, then $f(x) \in (\frac{q}{\alpha}, \frac{p}{\alpha})$.

Therefore for $x \in (a,b)$, $x \neq c$: $\frac{q}{\alpha} < f(x) < \frac{p}{\alpha}$

$$\Rightarrow \alpha \cdot \left(\frac{q}{\alpha}\right) > \alpha \cdot f(x) > \alpha \cdot \left(\frac{p}{\alpha}\right)$$

So that $\alpha \cdot f(x) \in (p,q)$.

$$\Rightarrow q > \alpha \cdot f(x) > p \Rightarrow p < \alpha \cdot f(x) < q$$

Theorem. The Constant Multiple Property. Suppose $\lim_{x \to c} f(x) = L$ and that $\alpha \in \mathbb{R}$. Then $\lim_{x \to c} \alpha \cdot f(x) = \alpha \cdot L$.

Proof. Suppose that (p,q) is an open interval about $\alpha \cdot L$.

Thus for any non-zero value of L, we meet the criterion given in the definition and thus we can conclude in those cases that

$$\lim_{x \to c} \alpha \cdot f(x) = \alpha \cdot L.$$

Case 3: $\alpha = 0$.

In this case, I usually show that the limit of a constant is that constant prior to the Constant Multiple Property (It extremely easy to show – ask for details).

But with that: $\lim_{x \to c} 0 \cdot f(x) = \lim_{x \to c} 0 = 0 = 0 \cdot L$ and thus this last case follows by invoking the "limit of a constant is that constant" property.

Extending This to Other Types of Limits.

The topological definition has analogs for the other type of limits as well.

For example:

Definition: $\lim_{x \to c^+} f(x) = L$ iff for all open intervals, (p,q), contained within the range, there is an open interval, (c,b), within the domain such that for all $x \in (c,b)$, $f(x) \in (p,q)$.

or

Definition: $\lim_{x \to c} f(x) = \infty$ iff for all $N > 0$, there is an open interval (a,b) within the domain with $c \in (a,b)$ such that if $x \in (a,b)$, $x \neq c$, then $f(x) > N$.

Example 2. Verify that $\lim_{x \to 0} \frac{1}{x^2} = 0.$

Let $N > 0.$

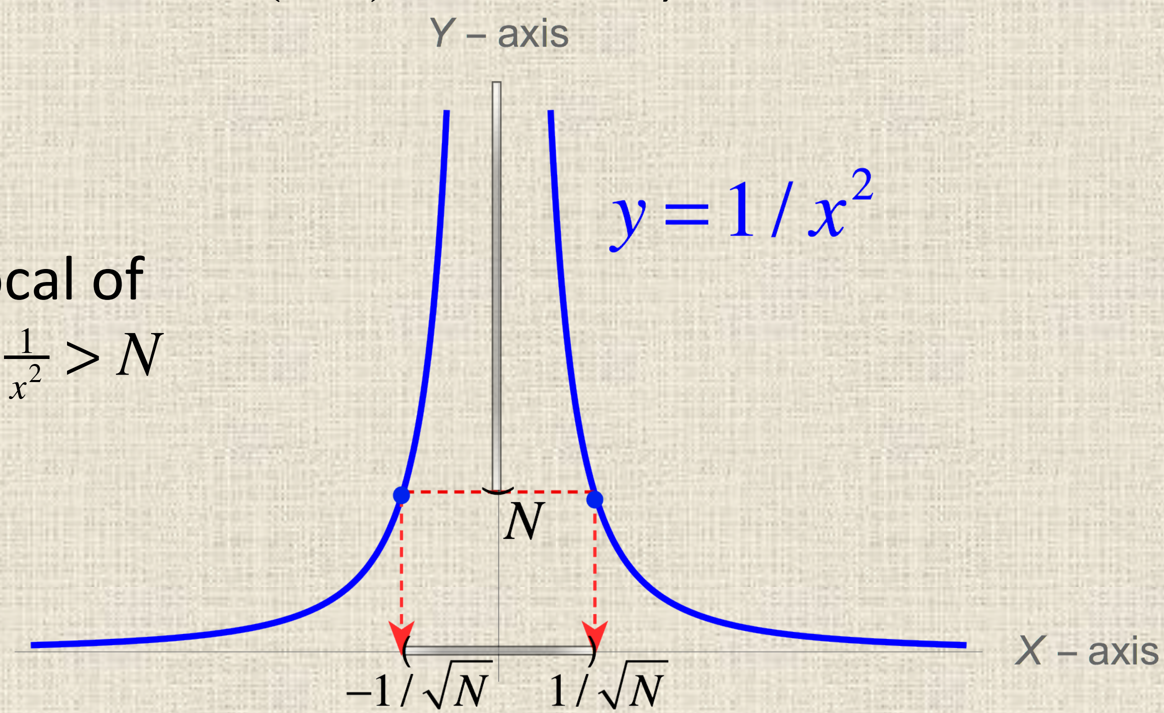
Then the inverse image of N are the two points $x = \frac{1}{\sqrt{N}}$ and $x = -\frac{1}{\sqrt{N}}.$

If we let $(a,b) = (-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ and if $x \in (a,b)$ with $x \neq 0$, then:

$$-\frac{1}{\sqrt{N}} < x < \frac{1}{\sqrt{N}}$$

$$\Rightarrow 0 < x^2 < \frac{1}{N}$$

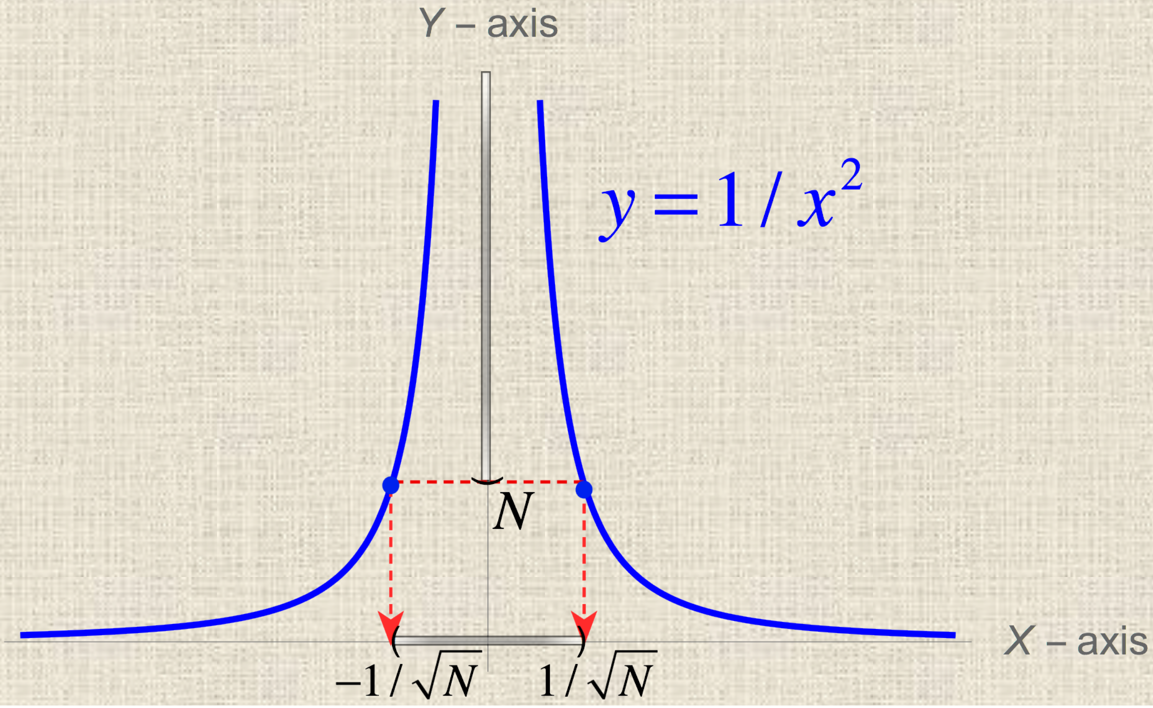
Then by taking the reciprocal of the right hand inequality: $\frac{1}{x^2} > N$



Example 2. Verify that $\lim_{x \to 0} \frac{1}{x^2} = 0.$

Thus given $N > 0$, there is an open interval about 0, $(-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$, such that if $x \in (-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ then $f(x) > N$.

Thus we can conclude by the topological definition that $\lim_{x \to 0} \frac{1}{x^2} = 0.$



Some Final Considerations.

The topological definition of limit can in many cases greatly simply the task of verifying specific limits and properties of limits.

However, often things based on higher level mathematics, such as topology, require a more concrete understanding of such topics as set theory.

In some cases, such as the proof of the property that the limit of a product is the product of the limits, the arguments boil down to basic case by case analyses.

In short some things can be more tedious in their articulation.

Some Final Considerations.

Even so, we think that the advantages of being able to tackle more complex limits than just limits of linear functions, as most do with the δ-ε definition, more than compensates for the added work required to use this definition in other situations.

Thus students can concentrate more on what the concept means instead of the drudgery of determining the right trick to get the argument to work.

The authors have used this in their elementary calculus course now for 5 or 6 years and the results have been very promising.

On the last final in Calculus 1 I gave, about 50% of students were able to articulate the proof that $\lim_{x \to 2} x^2 = 4$ while another 25% were able to set up the basic elements of that argument.

Some Final Considerations.

For those who are interested, a little more can be found in our article *"A Topological Definition of Limit for Use in Elementary Calculus"* in the September 2014 issue of the College Mathematics Journal, pg. 313.

For a more detailed exposition, including a proof of the equivalence of the topological definition to the δ-ε definition, contact the author at ccooper@uco.edu or mmcclendon@uco.edu and we'll send you our white paper on the subject.