Function Diagrams and Empty Parentheses

Kinetic Model of Functions

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Outline

Usual function computation mistakes

Empty parentheses as a partial solution

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Usual function computation mistakes

Example: Let $f(x) = 2x^2 - 3x + 5$. Find $f(x+2)$.

Usual Mistakes:

1. $f(x+2) = f(x) + 2 = 2x^2 - 3x + 5 + 2 = 2x^2 - 3x + 7$.
2. $f(x+2) = 2x^2 + 2 - 3x + 2 + 5 = 2x^2 - 3x + 9$.
3. $f(x+2) = 2(x+2)^2 - 3(x+2) + 5 =$ $2(x^2 + 4) - 3x - 6 + 5 = 2x^2 + 8 - 3x - 6 + 5 = 3x^2 - 3x + 7$.

It seems that the ideas of (1) x being a "dummy variable", and (2) that we can replace x with something else—particularly if that "something else" contains an x—can be very confusing to average students.

Note: Computing functions here is "kinetic," meaning there is are *actions* we perform on x, rather than making an "assignment" for each x, or identifying a set of ordered pairs.

Empty parentheses as a partial solution (not original!)

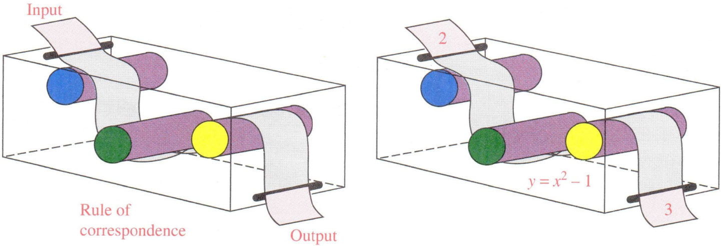
Basic Idea: Instead of the variable x to probe the action of f, use "empty parentheses" into which we can put our arguments.

Example: Let $f(\quad) = 2(\quad)^2 - 3(\quad) + 5$.

$$\begin{aligned}
f(1) &= 2(1)^2 - 3(1) + 5 = 2 \cdot 1 - 3 + 5 = 4, \\
f(-1) &= 2(-1)^2 - 3(-1) + 5 = 2 + 3 + 5 = 10, \\
f(-x) &= 2(-x)^2 - 3(-x) + 5 = 2x^2 + 3x + 5, \\
f(x+2) &= 2(x+2)^2 - 3(x+2) + 5 = 2(x^2 + 4x + 4) - 3x + 6 + 5 \\
&= 2x^2 + 8x + 8 - 3x - 6 + 5 \\
&= 2x^2 - 3x + 7,
\end{aligned}$$

Especially for more mature students who lack confidence but not wisdom or work ethic, this method seems to strike a chord.

Function diagrams in current texts (especially College Algebra)



“Kinetic” in the sense something is happening, but not realistic. “Rule of correspondence” harkens to the more abstract (correct?) definitions of function. (Gustafson & Frisk) input 4 input 4

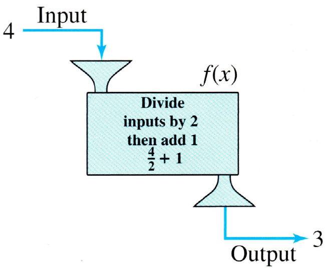
$$f(x) = \frac{x}{2} + 1$$

$$f(4) = \frac{4}{2} + 1$$

$$= 2 + 1$$

$$= 3$$

Figure 2.35



Somewhat more “kinetic,” but not ideal for visually tracing the actions on the input. Does relate the formula to the “process.” (Coburn)

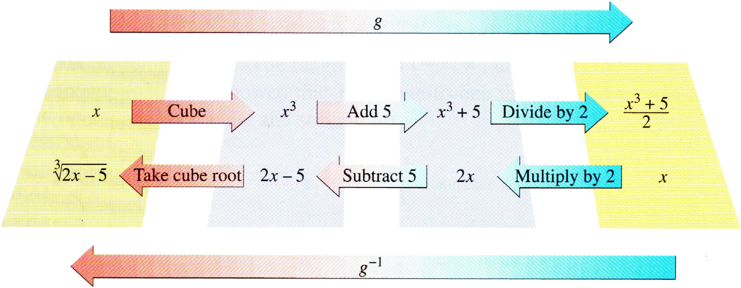
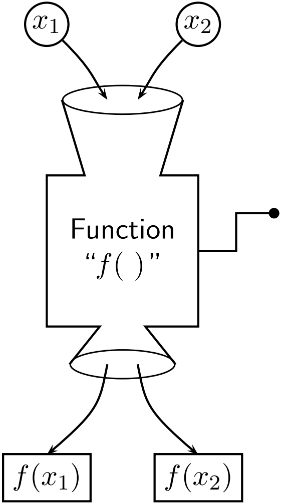


Figure 2.62

More "kinetic," and shows something of function composition and inverses. Hints at diagrams between domain and range. Illustrates how some invertible functions f can be inverted in stages, if f is itself a series of invertible steps (unlike, say, linear fractional transformations which are more convoluted). (Dugopolski)

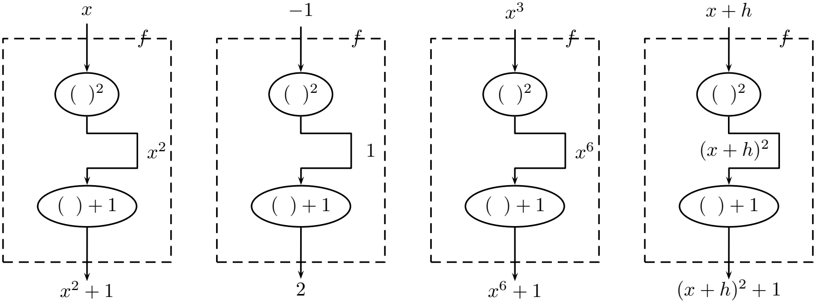
My early attempts at function diagrams at the board



- These are fine for textbook illustrations.
- Not so good for drawing many of them on the board.
- Particularly for a "chain" of functions.
- Out of necessity, settled on a combination of "empty parentheses" and ovals, rectangles—simple figures:

Flow chart-like diagrams: “kinetic” model of functions

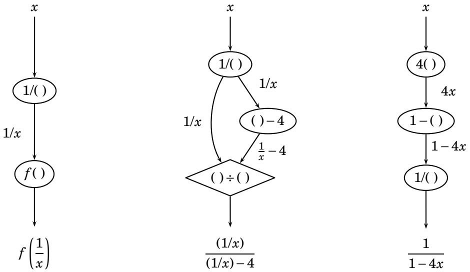
Example: $f(\quad) = (\quad)^2 + 1.$



- $f(x) = (x)^2 + 1 = x^2 + 1$
- $f(-1) = (-1)^2 + 1 = 1 + 1 = 2.$
- $f(x^3) = \left(x^3\right)^2 + 1 = x^6 + 1.$
- $f(x+h) = (x+h)^2+1 = (x+h)(x+h)+1 = x^2+2xh+h^2+1.$

Useful for domain computations

Example: Suppose $f(x) = \frac{x}{x-4}$. Then $f\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right) - 4}$.

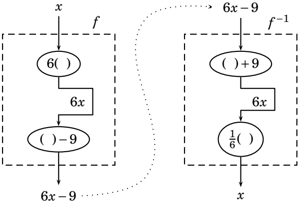


Domain: $\mathbb{R} - \left\{0, \frac{1}{4}\right\}$. (Where nothing goes wrong!)

- $x \neq 0$ is clear from the first two models; glossed over in third.
- $x \neq \frac{1}{4}$ is clear from the second or third models.
- For most x, the second and third models are the same, but only the second shows how x is truly processed.

Diagram of simple function inversion

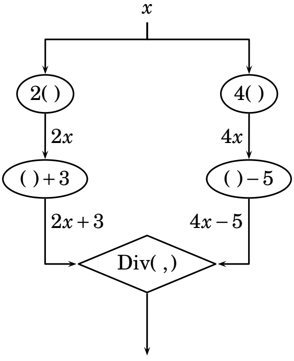
- With dummy variables:
  - $f(x) = 6x - 9$
  - $f^{-1}(x) = \frac{1}{6}(x + 9)$
- With empty parentheses:
  - $f(\quad) = 6(\quad) - 9$
  - $f^{-1}(\quad) = \frac{1}{6}((\quad) + 9)$
- Using a function diagram:



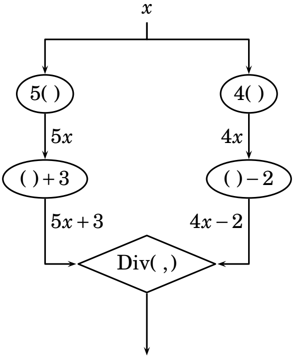
Linear fractional transform and inverse example

$$f(x) = \frac{2x+3}{4x-5}$$

Here it is not so obvious how to invert the function, and we have to resort to *algebra*: solve $y = f(x)$ for x, or switch the variables and solve for y. Either way yields the form of $f^{-1}(\quad)$.



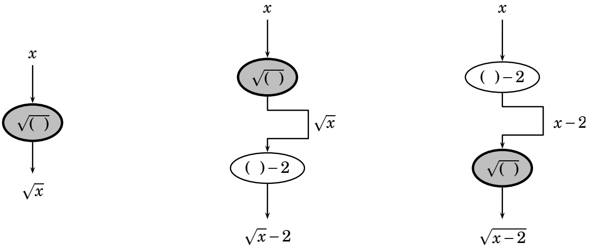
$f(x) = \frac{2x+3}{4x-5}$

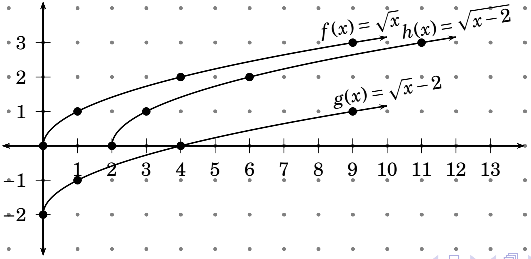


$f^{-1}(x) = \frac{5x+3}{4x-2}$

Transformations of known functions

Example: $f(\quad) = \sqrt{(\quad)}$. “Main function” in gray.

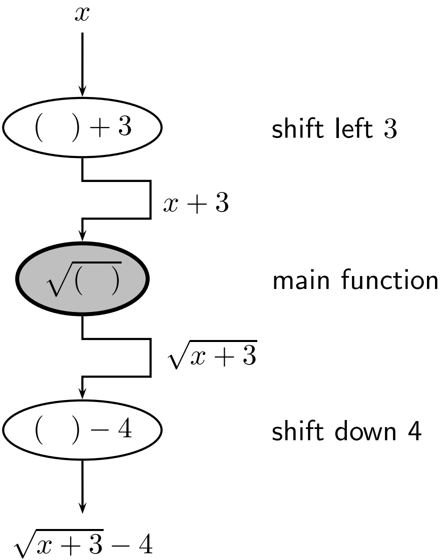




Chasing through transformations, example

$$y = f(x) = \sqrt{x+3} - 4.$$

- Transformations that occur **before** the main function affect the input variable, and thus some horizontal aspect of the graph.
- Transformations that occur **after** the main function affect the output variable, and thus some vertical aspect of the graph.



x

$(\quad) + 3$ shift left 3

$x + 3$

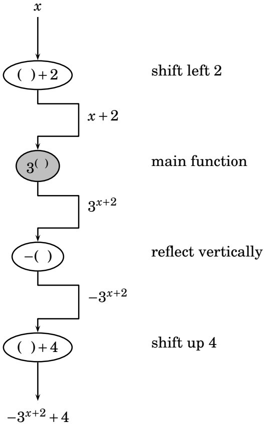
$\sqrt{(\quad)}$ main function

$\sqrt{x+3}$

$(\quad) - 4$ shift down 4

$\sqrt{x+3} - 4$

Example: $f(x) = -3^{x+2} + 4$.



x

$(\ )+2$ shift left 2

$x+2$

$3^{(\ )}$ main function

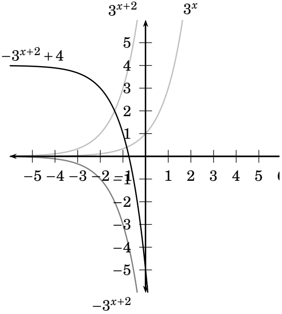
3^{x+2}

$-(\ )$ reflect vertically

-3^{x+2}

$(\ )+4$ shift up 4

$-3^{x+2}+4$

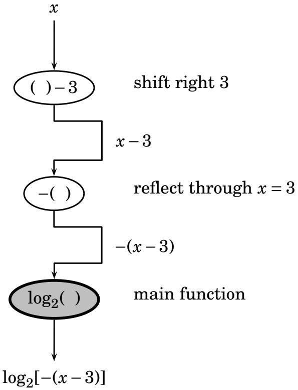


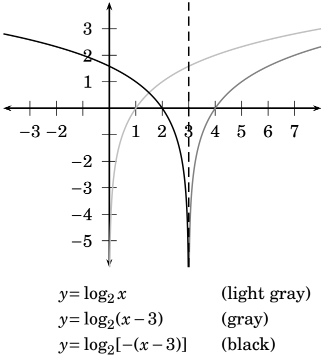
Of course, with *linear* transformations of the input or output variable, the basic shape stays the same and it may be easier to identify features such as asymptotes, and plot some important points to make a quick sketch.

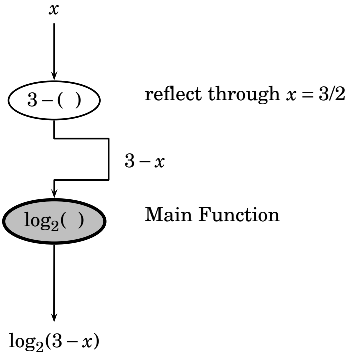
Reflections around other vertical/horizontal lines

Example: $f(x) = \log_2(-(x-3)) = \log_2(3-x)$.

- Can take $y = \log_2 x$, move three to right, reflect horizontally through the line $x = 3$.
- Can take $y = \log_2 x$, reflect horizontally through line $y = 3/2$.







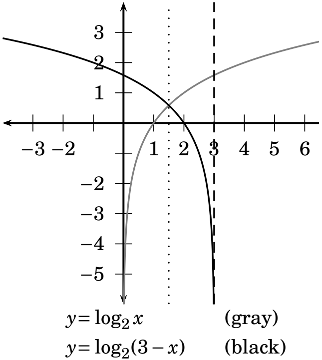
x

$3-(\ )$ reflect through $x = 3/2$

$3-x$

$\log_2(\ )$ Main Function

$\log_2(3-x)$



$y = \log_2 x$ (gray)

$y = \log_2(3-x)$ (black)

General remarks about function diagrams and transformations

To use diagrams for linear transformations of the input or output varibles of a function, it is best to have forms such as:

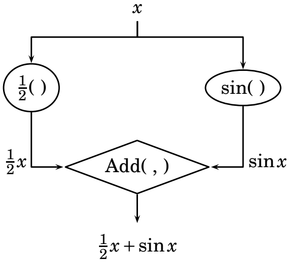
$$\begin{aligned} y &= Af(B(x-h))+k && \text{(reflected if } A \text{ or } B \text{ is negative)} \\ y &= K - Af(B(x-h)) && \text{(reflected through } y = K/2\text{)} \\ y &= Af(B(H-x))+k && \text{(reflected through } x = H/2\text{)} \end{aligned}$$

These are preferable to $y = Af(Bx + C) + D$, which should be rewritten into one of the forms above. Diagrams constructed naturally as one would parse y for an arbitrary x.

I'm happy to send pages that develop this in more detail.

Calculus: "Addition Rule"

Example: $f(x) = \frac{1}{2}x + \sin x$



x

$\frac{1}{2}(\,)$

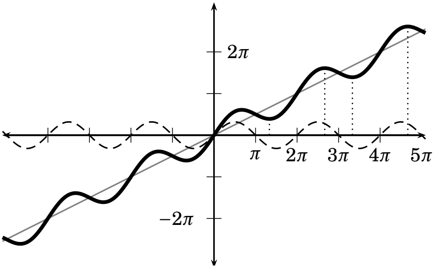
$\sin(\,)$

$\frac{1}{2}x$

Add(,)

$\sin x$

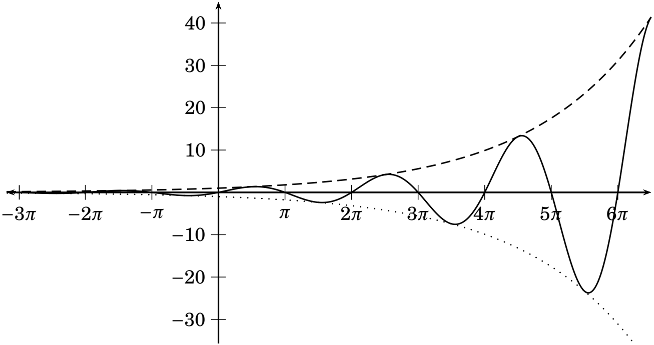
$\frac{1}{2}x + \sin x$



Are we adding $\frac{1}{2}x$ to the $\sin x$ function, or vice-versa?

Calculus: "Product Rule"

Example: $f(x) = 1.2^x \sin x$



Are we multiplying 1.2^x by $\sin x$, or vice-versa? (Easier to think of 1.2^x as the "amplitude" in some sense.)

Conclusions (opinions)

1. Some students want to see the *action* of the function: what does it do to the input to yield the output?
2. Perhaps it is better (for some students) to begin with “kinetic” ideas of functions (as actions on the variables), and slowly introduce function graphs, functions as assignments or mappings (good EFI term!). Eventually explain why $f(x) = (x+1)^2$ is really the same function as $f(x) = x^2 + 2x + 1$. (Functions as black boxes?)
3. Leave out “relations?” Implicit functions are a stand-alone.

4. Eventually the mechanisms can be read from the formula.
5. Following variables through diagrams may be easier than through a formula, at first.
6. Most texts use some diagrams, but do not develop them. There may be some low-lying fruit here.
7. On the other hand, there are technicalities to work through.
8. Next Up: Assigning diagram construction to students, to see if it helps. ("Nothing But Liebniz" in computing derivatives helped most of my students in Calculus 1. More visual aids may be useful with functions too.)