An Interesting Student-Posed Question in Linear Algebra

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In our existing Linear Algebra course, we currently use the Gareth Williams text, in which our fairly typical semester looks like:

1. Linear Systems
2. Matrices and Inverses
3. Determinants
4. General Vector Spaces
5. Eigenvalues and Eigenvectors
6. Linear Transformations

It was in a section of Linear Algebra that I was teaching during the fall 2012 semester, and near the end of the term that we reached the topic of *linear transformations*. I proceeded to introduce them to the typical definition of a linear transformation:

(U,+, .) and (V,+, .) are vector spaces, let **u**,**v** be in U and let k be a (real) scalar.
L: (U,+, .) → (V,+, .) is a linear transformation if:

1. L(**u** + **v**) = L(**u**) + L(**v**)
2. L(k **u**) = kL(**u**)

As most introductory Linear Algebra classes would, we proceeded to investigate examples of transformations to decide/prove if they were linear, or non-linear. It was in the midst of this investigation that a curious student posed the central question of this paper:

"*Can we find an example of a transformation that satisfies one of the two properties but fails on the other property?*"

Once I understood that it was truly asked out of an inquisitive nature (and not as a *time waster* during class discussion), I really started to think seriously about his question. I first thought it would be an easy thing to answer but in the few moments during class that we could give to it, nothing worked and nothing was obvious. So, along with their normal homework at this stage of the semester, I'd asked that they also go home and try to think of their own examples. Since this was a very late part of our semester, they didn't all have time for this *extra inquiry*…but those that did were unsuccessful. To this point, I had limited time to devote to the student's question but was unsuccessful in obtaining a solution of my own for the next class period. Getting a little frustrated but needing to finish the requisite content, we pushed it aside with my gentle promptings to keep it on the back burner but in active thought where time permited; definitely to let myself and the class know if anyone was successful in finding an example.

In preparation for finals, I threw out the student's question to my friends on the Project NExT email list. Becoming a Project NExT fellow in the summer of 1997, I quickly found this as one of the most useful resources one could tap into. It consists of fellow PhD teachers of mathematics from around the country, and the questions and topics posed on the list range considerably. The reactions I got back were quite surprising to me, because usually such inquiries are flooded with responses that exactly fit the situation. I heard feedback like:

"*we've discussed this situation numerous times on the list, and I don't think we've ever come to a satisfying conclusion…*"

This is when I didn't feel quite so badly on not having an immediate, or even semi-immediate response for my student, figuring it wasn't quite the common issue/question that I figured it might be nor so obvious on what transformation would work. But one personally directed response came to me, that provided some hope! He told me that fresh out of graduate school, he was teaching a section of Linear Algebra in which a student posed this exact question. He wasn't sure of an example so he wrote back to his dissertation advisor, and was subsequently given one that satisfied this student's inquiry and he *filed it away for safe keeping*. He'd indicated on not being sure exactly where the example had been tucked away but would look for it and get back with me… thankfully he found it!!! Here is that example:

Consider $(\mathbf{x},\mathbf{y})$ in R^2 and a scalar k, let $L:(R^2,+,\,.\,) \rightarrow (R,+,\,.\,)$ be defined by:

$$L(\mathbf{x},\mathbf{y}) = \begin{cases} \mathbf{x}, & \text{if } \mathbf{y} \text{ is } 0 \\ 0, & \text{if } \mathbf{y} \text{ is not } 0 \end{cases}$$

This was exactly an example of the type that we were searching for and here is a brief outline of the argument to showed it satisfied the second property but failed at the first.

1. $L(k(\mathbf{x},\mathbf{y})) = L(k\mathbf{x},k\mathbf{y})$
    a. If $\mathbf{y}$ is 0 then $k\mathbf{y} = 0$ and $L(k\mathbf{x},k\mathbf{y}) = k\mathbf{x} = kL(\mathbf{x},\mathbf{y})$
    b. If $\mathbf{y}$ is not 0 then in general, $k\mathbf{y}$ is not 0 and $L(k\mathbf{x},k\mathbf{y}) = 0 = kL(\mathbf{x},\mathbf{y})$
2. $L(1,0) + L(0,1) = 1 + 0 = 1$
   but… $L(1,1) = 0$ (since $\mathbf{y}$ is 0)

I am sure other such examples exist, but since it was definitely not obvious to me at the time (nor obvious to almost all on the expansive Project NExT list either), and I wanted to speak on this issue should it come up again. If normal examples exist which posses the first property of a linear transformation but fail property two, those examples escape my attention at this stage. Whatever example might exist, I believe it would have to be *discontinuous* in nature like the above example. It's my hope that this example can not only satisfy your curiosity but also help if such a question is posed in one of your classes. But don't get me wrong, I don't think classroom investigation and inquiry is a bad thing at all, and I believe a great deal can be gained when our students can see that the answer to question posed is not always *immediately obvious*; even or especially to their professor.