HORNER'S ALGORITHM AND MOESSNER'S PROPERTY OF NATURAL NUMBERS

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THIS PRESENTATION IS BASED ON "A NOTE ON AN ADDITIVE PROPERTY OF NATURAL NUMBERS," BY J VAN YZEREN IN THE JANUARY 1959 ISSUE OF THE AMERICAN MATHEMATICAL MONTHLY.

First, a property of the natural numbers attributed to Alfred Moessner's (1951) is illustrated below.

First, for n = 3.

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,…

1,3, 7,12, 19,27, 37,48, 61,75, 91,108, 127,147, 169,192, 217,243, 271,300, 331

1 8 27 64 125 216 343 512 729 1000 1331

The second line is obtained from the first line by forming partial sums after eliminating every 3rd term. The third line is then the partial sums of the second line obtained by eliminating every 2nd term. The resulting sequence is just 1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3, 8^3, 9^3, 10^3, 11^3,…

For n = 4:

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,…

1,3,6, 11,17,24, 33,43,54, 67,81,96, 113,131,150, 171,193,216, 241,267,294 323,353,384

1,4, 15,32, 65,108, 175,256, 369,500, 671,864, 1105,1372, 1695,2048

1 16 81 256 625 1296 2401 4096

In this case each line is the sequence of partial sums of the one above by eliminating every 4th term, then every 3rd term, and finally every 2nd term. The sequence in the last line is just 1^4, 2^4, 3^4, 4^4, 5^4, 6^4, 7^4, 8^4,…

Now, for n = 5:

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,…

1,3,6,10, 16,23,31,40, 51,63,76,90, 106,123,141,160, 181,203,226,250, 276,303,331,360, 391

1,4,10, 26,49,80, 131,194,270, 376,499,640, 821,1024,1250, 1526,1829,2160 2551

1,5, 31,80 211,405, 781,1280, 2101,3125, 4651,6480, 9031

1, 32, 243, 1024, 3125, 7776 16807

Following the pattern above each line is the partial sums of the one above after eliminating every 5th term, every 4th term, every 3rd term, and finally every 2nd term and the bottom line is just $1^5, 2^5, 3^5, 4^5, 5^5, 6^5, 7^5$,... Hence, Moessner's Property states that for any natural number n one can construct n sequences of partial sums from the sequence of natural numbers by first eliminating every nth term, then every (n-1)th term from that resulting sequence, then every (n-2)th term, etc. until one finally obtains a sequence of partial sums by eliminating every 2nd term which is just the sequence of nth powers of the natural numbers.

A proof of this result can be structured on Horner's Algorithm for expressing an arbitrary polynomial in powers of (x-1), (x-2), (x-3), …, or (x-n) for any natural number n. For example, given the general polynomial of degree 3: $a_0x^3+a_1x^2+a_2x+a_3$, then the coefficients of the corresponding polynomial in powers of (x-1) are found in the following way:

a_0	a_1	a_2	a_3
	a_0	a_0+a_1	$a_0+a_1+a_2$
a_0	a_0+a_1	$a_0+a_1+a_2$	$\underline{a_0+a_1+a_2+a_3}$
	a_0	$2a_0+a_1$	
a_0	$2a_0+a_1$	$\underline{3a_0+2a_1+a_2}$	
	a_0		
$\underline{a_0}$	$\underline{3a_0+a_1}$		

The underlined terms are the coefficients of the new polynomial:
$a_0(x-1)^3 + (3a_0+ a_1)(x-1)^2 + (3a_0+2a_1+a_2)(x-1) + (a_0+a_1+a_2+a_3)$

For example, given x^3+x^2+x+1, the above becomes

1	1	1	1
	1	2	3
1	2	3	$\underline{4}$
	1	3	
1	3	$\underline{6}$	
	1		
$\underline{1}$	$\underline{4}$		

which gives $(x-1)^3+4(x-1)^2+6(x-1) + 4 = x^3 + x^2 + x + 1$.

Applying Horner's Algorithm to <u>this</u> gives the expansion of x^3+x^2+x+1 in powers of (x-2):

1	4	6	4
	1	5	11
1	5	11	<u>15</u>
	1	6	
1	6	<u>17</u>	
	1		
<u>1</u>	<u>7</u>		

which gives $(x-2)^3+7(x-2)^2+17(x-2)+15 =(x-1)^3+4(x-1)^2+6(x-1)+4 = x^3+x^2+x+1$.

The algorithm can be extended to arbitrary polynomials.

J. Van Yzeren's note gives a proof of Moessner's property for each choice of n by applying Horner's Algorithm to the polynomial x^n. The first block gives x^3 in powers of (x-1), the second block in powers of (x-2) and the third block in powers of (x-3):

1	0	0	0	1	3	3	1	1	6	12	8
	1	1	1		1	4	7		1	7	19
1	1	1	<u>1</u>	1	4	7	<u>8</u>	1	7	19	<u>27</u>
	1	2			1	5			1	8	
1	2	<u>3</u>		1	5	<u>12</u>		1	8	<u>27</u>	
	1				1				1		
<u>1</u>	<u>3</u>			<u>1</u>	<u>6</u>			<u>1</u>	<u>9</u>		

Hence, $x^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 = (x-2)^3 + 6(x-2)^2 + 12(x-2) + 8$
$= (x-3)^3 + 9(x-3)^2 + 27(x-3) + 27$ from which it is clear that the constant terms are just $1^3, 2^3, 3^3$.

Clearly, this process can be continued to produce the coefficients for an expansion of x^3 in powers of (x-n) for any natural number n. Suppressing all the copied figures (appearing in the copied first rows but also in all the even rows), and then placing the blocks underneath each other yields:

$x^3 + 0x^2 + 0x + 0$

1	0	0	0
1	1	1	1
1	2	3	
1	3		
1	4	7	8
1	5	12	
1	6		
1	7	19	27

$x^4 + 0x^3 + 0x^2 + 0x + 0$

1	0	0	0	0
1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1	5	11	15	16
1	6	17	32	
1	7	24		
1	8			
1	9	33	65	81

The second block above is the analogue for x^4. Apparently, these arrays contain Moessner's sequences for n=3 and n=4 written vertically. According to Horner's algorithm the underlined number are the constant terms in the expression for x^3 or x^4 in powers of (x-1), (x-2), (x-3), etc. and the constant terms must equal $1^3, 2^3, 3^3, \ldots$ for n=3 and $1^4, 2^4, 3^4, \ldots$ for n=4. Clearly, this construction can be generalized to x^n to obtain a proof of Moessner's Property for any n.

For n = 2 Moessner's Property leads to a fairly well-known result about natural numbers:

1,2,3,4,5,6,7,8,9,10,…

1,4,9,16,25,36,49,64,81,100,…

In other words, summing the odd integers gives the sequence of squares of natural numbers.