Romeo, Romeo, What Art Thou Differential Equations?

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Abstract

Following the ideas of Steven Strogatz in "*Love Affairs and Differential Equations*"[1], we attempt to model the dynamics of romantic relationships using differential equations. In particular, we investigate the behavior of our two functions "Romeo" and "Juliet" based upon their personality types, considering two cases: i) when a member's feelings are affected only by their partner's feelings (which produces exact solutions), and ii) when a member's feelings are affected by both their partner's feelings and their own feelings (which requires an RK approximation). By looking at variations in both their personality parameters and their initial feelings, we hope to make some general statements about the expected evolution and outcome of such relationships. We also make a brief comparison to a real-life collegiate survey, and predict the results for an "average" boy-girl relationship.

Love may be a many-splendored thing, but it is also highly complicated and seemingly unpredictable. Just ask any Shakespearean lover about its twist and turns! Perhaps all of us, at one time or another, have pondered about how to proceed in love, and have longed for a future look at the fate of our romantic decisions. Mathematically, this leaves us calling out from the balcony for differential equations to speak to us! Stephen Strogatz famously searched for such dynamical models in his article: *Love Affairs and Differential Equations*"[1], and also in his text "Non-Linear Dynamics and Chaos"[2]. We follow the climb and analyze his model further, hoping to help our star-crossed lovers, and perhaps even help ourselves in the game of love.

Model I

We begin our attempt with a very simple model of romantic relationships, which assumes that a person's feelings are only affected by the feelings of the other. To represent such quantities with variables, we define:

$$R(t) = \text{Romeo's feelings}$$
$$J(t) = \text{Juliet's feelings}$$

which will be measured on a scale from -1 to1, with negative values representing a level of dislike, and positive values representing a level of affection. To design an appropriate system which describes how their feelings change over time, we then propose the differential equations:

$$R'(t) = a\,J(t) \tag{1}$$
$$J'(t) = b\,R(t) \tag{2}$$

where the constants a, b denote “commitment factors”. A positive constant indicates that the other person’s affection entices them, and increases their own feelings. A negative constant, on the other hand, indicates that the other person’s affection repels them, and measures their fear of commitment.

For the task of solving these equations, we first take the second derivative of equation (1) to obtain:

$$R''(t) = a\, J'(t) \tag{3}$$

which can be combined with equation (2) to change our system into a single 2nd-order differential equation:

$$R''(t) = a\, b\, R(t). \tag{4}$$

This has a well-documented solution method [3], which involves guessing the generic solution $R = e^{rt}$ with unknown constant r. We then calculate the subsequent derivatives, $R'(t) = r\, e^{rt}$ and $R''(t) = r^2\, e^{rt}$, to substitute back into our DE (4), and solve for r:

$$\begin{aligned} &R''(t) = a\, b\, R(t) \\ &\Rightarrow r^2\, e^{rt} = a\, b\, e^{rt} \\ &\Rightarrow r^2 = a\, b \\ &\Rightarrow r = \pm\sqrt{(ab)}. \end{aligned} \tag{5}$$

This sets the stage for finding exact solutions and revealing the possible fates of our potential lovers.

Model I, Case 1: The first scenario for Model I involves a and b with the same sign (both positive or both negative), which gives us real roots for r in equation (5). This would represent two members which are either both responsive to affection or both repelled by it. Such real roots placed inside our guess $R(t) = e^{rt}$ produces the combined fundamental solutions for Romeo’s feelings:

$$R(t) = c_1\, e^{\sqrt{(ab)}\, t} + c_2\, e^{-\sqrt{(ab)}\, t}. \tag{6}$$

Taking the derivative of (6) and combining with equation (1) then gives us the corresponding solution for Juliet’s feelings, namely:

$$J(t) = c_1\, [\sqrt{(ab)}/a]\, e^{\sqrt{(ab)}\, t} - c_2\, [\sqrt{(ab)}/a]\, e^{-\sqrt{(ab)}\, t}. \tag{7}$$

Notice that the first term $e^{\sqrt{(ab)}\, t}$ in each solution (6) and (7) embodies a growth factor, which runs away in time towards positive or negative infinity, depending on the sign of its coefficient. On the other hand, the second term $e^{-\sqrt{(ab)}\, t}$ represents a damping factor, which asymptotically approaches 0 and thus has no long-term effect. Therefore, the futures of both Romeo and Juliet depend solely on the coefficients of their first term. Romeo, for example, has the following possibilities in his future feelings toward Juliet:

$c_1 > 0$ means a positive growth factor, so R(t) increases to ∞
$c_1 < 0$ means a negative growth factor, so R(t) decreases to $-\infty$
$c_1 = 0$ means no growth factor, so there is only a damping term, and R(t) approaches 0.

Juliet has a similar fate, except that her coefficient involves the factor $c_1\, [\sqrt{(ab)}/a]$ instead of c_1. Note that Romeo’s and Juliet’s fates are definitely related, but depending upon the signs of a and b, they are not necessarily the same.

So, finding the final outcomes for our lovers depends upon finding c_1, which can be done by evaluating the initial conditions t = 0 in equations (6) and (7), namely:

$$R(0) = c_1 + c_2$$
$$J(0) = c_1 [\sqrt{(ab)}/a] - c_2 [\sqrt{(ab)}/a]. \quad (8)$$

Rearranging and combining these, we determine the constant we are interested in:

$$c_1 = J(0)\, a / 2 \sqrt{(ab)} + R(0)/2. \quad (9)$$

As we have seen, the sign of c_1 determines Romeo's romantic future, so setting $c_1 = 0$ defines his "tipping point" of success. Solving for Romeo's initial condition, we achieve

$$R(0) = - J(0)\, a / \sqrt{(ab)}, \quad (10)$$

which establishes the boundary of whether his feelings evolve into love or hate. Therefore, Romeo's feelings have three possible outcomes, depending on their initial states and commitment factors:

$R(0) > - J(0)\, a / \sqrt{(ab)}$ => Eternal Love
$R(0) < - J(0)\, a / \sqrt{(ab)}$ => Eternal Hate
$R(0) = - J(0)\, a / \sqrt{(ab)}$ => Eternal Apathy

Juliet has the exact same tipping point, but depending on the signs of a and b, she may have exactly the same or opposite romantic destiny.

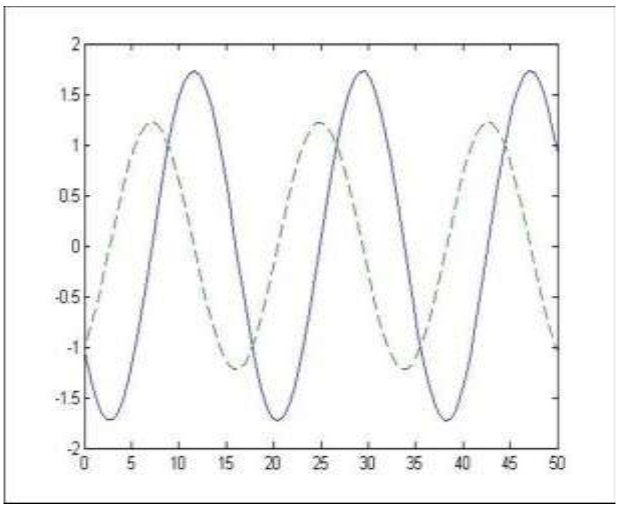
Now we are ready to map out all the possible outcomes for our pair of lovers in Model I, Case I. The results for each scenario are summarized in the table below:

Signs of a,b	**Initial Conditions**	**Romantic Fate**
$a,b > 0$	$R(0)$ and $J(0) > 0$ (both start with like)	Love connection.
(both respond to commitment)	$R(0)$ and $J(0) < 0$ (both start with dislike)	Hate connection
	$R(0) > 0, J(0) < 0$ (begins with like/dislike):	
	i) $R(0) > - J(0)\, a / \sqrt{(ab)}$	Love connection
	ii) $R(0) < - J(0)\, a / \sqrt{(ab)}$	Hate connection
	iii) $R(0) = - J(0)\, a / \sqrt{(ab)}$	Mutual apathy
$a,b < 0$	$R(0)$ and $J(0) > 0$ (both start with like)	
(both afraid of commitment)	i) $R(0) > - J(0) a / \sqrt{(ab)}$	Love/Hate
	ii) $R(0) < - J(0)\, a / \sqrt{(ab)}$	Hate/Love
	iii) $R(0) = - J(0)\, a / \sqrt{(ab)}$	Mutual Apathy
	$R(0)$ and $J(0) < 0$ (both start with dislike)	
	i) $R(0) > - J(0)\, a / \sqrt{(ab)}$	Love/Hate
	ii) $R(0) < - J(0)\, a / \sqrt{(ab)}$	Hate/Love
	iii) $R(0) = - J(0)\, a / \sqrt{(ab)}$	Mutual apathy
	$R(0) > 0, J(0) < 0$ (begins with like/dislike)	Love/Hate
	$R(0) < 0$ and $J(0) > 0$ (begins with dislike/like)	Hate/Love

Model I, Case 2: The second case to investigate in Model I is when a and b have opposite signs (a is positive and b is negative, or vice versa). Now our resulting solution root $r = \pm \sqrt{(ab)}$ is a complex number, creating a different combination of fundamental solutions:

$$R(t) = c_1 \cos(\sqrt{(ab)}\, t) + c_2 \sin(\sqrt{(ab)}\, t)$$
$$J(t) = - c_1[\sqrt{(ab)}/a] \sin(\sqrt{(ab)}t) + c_1[\sqrt{(ab)}/a] \cos(\sqrt{(ab)}t). \qquad (11)$$

These sine and cosine solutions are periodic, which reveals that our lovers will face an endless rollercoaster of pursuit and retreat. Alas! We see an example below:



The frequency of these cycles will be $f = \sqrt{(ab)}$, with an amplitude determined by initial conditions R(0) and J(0). However, there may be some good news, depending on your perspective: since the feelings of each are the derivative of the other (with a possible sign difference), one will be out of phase by $\pi/2$. They will both be in love exactly ¼ of the time!

Summary of Model I

We close our analysis of Model 1 with a summary of broad, general statements that may have some useful relevance and guidance in real relationships. They include:

- If even one member is afraid of commitment, there can never be a love connection.
- Love, Hate, and Love/Hate can last eternally.
- Hate can turn into love if the other's love is strong enough.
- If both members respond well to commitment, they will always reach the same conclusion.
- If the members respond oppositely to commitment, there will be a "cat and mouse" game forever.
- Both can be happy in a romantic chase for 25% of the time.

We can see that our first attempt at modeling love suggests both caution and hope for lovers. However, this model is very simplistic, and certainly can not address the true complexities of love. It calls from the balcony for a more sophisticated model.

Model II

Our more advanced model proposes that a person's feelings are not only affected by the other, but also by the current state of their own feelings. When we add this additional component, our equations take the form:

:

$$R'(t) = rr\, R(t) + rj\, J(t) \quad (12)$$

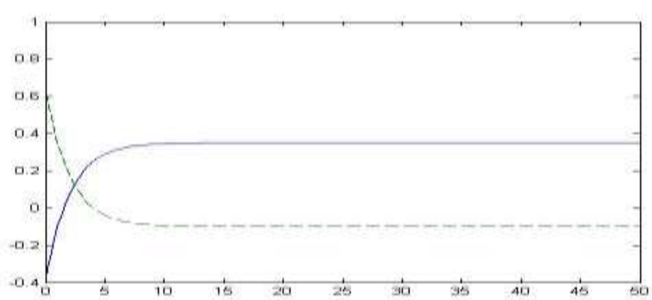
$$J'(t) = jr\, R(t) + jj\, J(t) \quad (13)$$

where rr, rj, jr, and jj are constants. The pair of constants rj and jr take the same role as a and b in Model I, namely as parameters registering each member's attraction or fear towards commitment. The additional pair, rr and jj, are new constants reflecting how a person is influenced by their own feelings. A positive value reflects a continued confidence in how they are feeling, while a negative reveals a hesitation and tendency to change their mind.

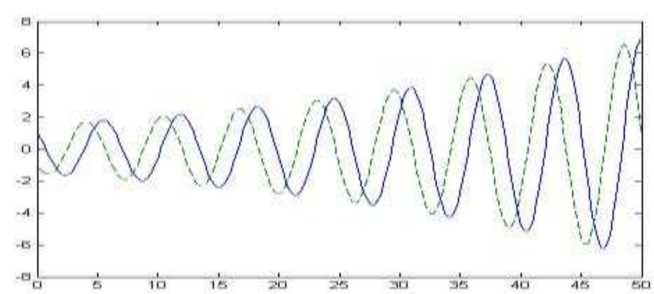
Unfortunately, adding this new complexity makes general, exact solutions elusive. They produce a set of non-linear differential equations which require approximation methods. Our choice to continue the investigation was to rely upon MatLab's ODE solver[4] which utilizes a Runge-Kutta approximation scheme. Because of the increased complexity, we decided to focus on a few key questions, rather than present a comprehensive study.

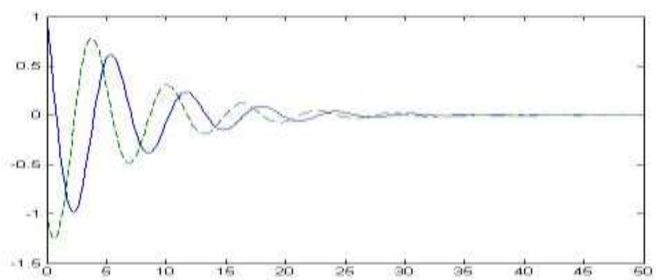
Question 1: Model I saw the only possible fates as eternal love, eternal hate, eternal love/hate, eternal apathy, and endless pursuit cycles. Does Model II contain any other possible endings?

We found out that, yes, other possible types of endings do exist. For example, precisely setting the constants: rr = -jr and rj = -jj produces possible final states besides 0 or $\pm\infty$. In fact, any pair of final values can be achieved. We have shown an example below with rr = -jr =.2 and rj = -jj = .7:



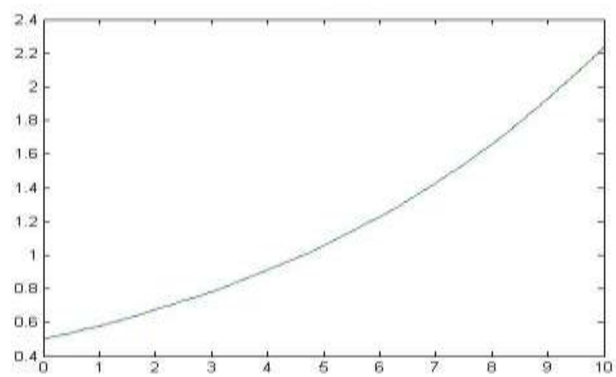
Another possible ending type would be when rj = - jr, which gives pursuit cycles that are increasing or decreasing over time. Below, examples are set at rr = -.12, rj = 1,jj = -.18, jr = -1, and rr = -.12, rj = 1, jj = .18, jr = -1, respectively:





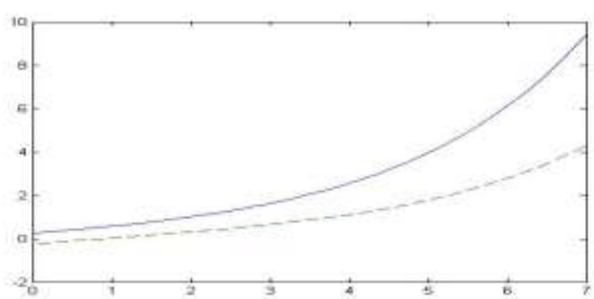
Question 2: Should you look for a romantic partner that is just like you?

This question requires setting rr = jj, rj = jr, and R(0) = J(0), making our lovers identical in personality and initial feelings. We find that if Romeo and Juliet begin in love, and their confidence outweighs their fear of commitment (or vice versa), then a love connection is produced! In terms of actual values, love requires R(0),J(0)>0, and rr + rj >0, and jj + jr >0. The fate graphed below corresponds to rr = jj = -.35, rj = jr = .5:



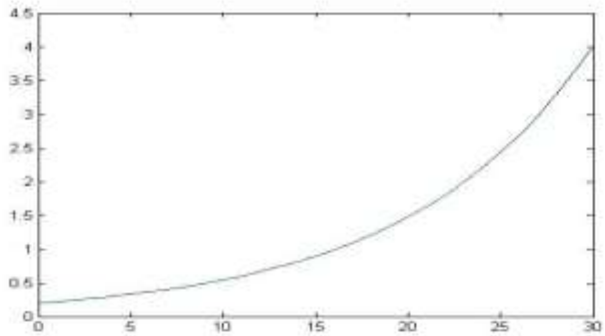
Question 3: Should you look for someone who is exactly the opposite of you?

To see these results we are going to set rr = - jj, rj = - jr, and R(0) = - J(0). And, yes, if Romeo starts out loving Juliet, and if Romeo's confidence is greater than his fear of commitment, then a love connection does occur. Specifically, it requires R(0)>0, rr>0, rj<0, and |rr| >|rj|. The graph below was set with rr = -jj = .66 and rj = -jr = -.5:



Question 4: In Model I, we found that if even one member is afraid of commitment, there can never be a love connection. Is this still the case?

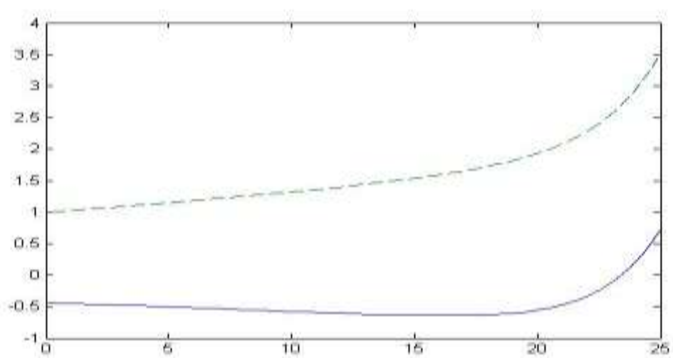
We find that if lovers are both equally scared, equally more self-confident, and equally in love, then they can indeed have a romantic future together. It works when R(0)=J(0)>0, rr = jj, rj = jr, and rr > |rj|. A specific example, rr = jj = .5, rj = jr = -.4, is displayed below:



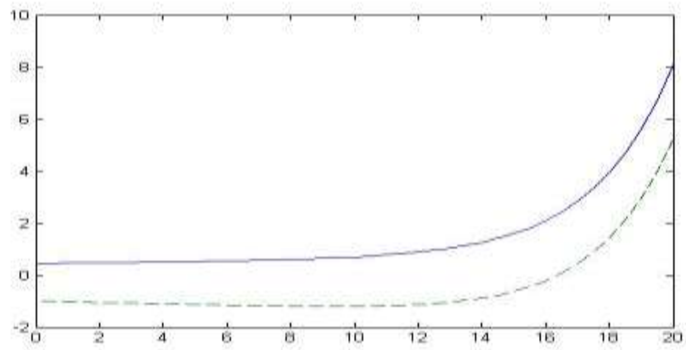
Question 5: Now for the question many have been waiting for! Assuming that you are involved in an average relationship, what can you expect for your love life?

We attempted some form of an answer by giving a collegiate survey to estimate an average set of constants for boys and girls. The set of questions composing the survey tried to quantify the commitment and confidence factors possessed by each gender. They included self-assessment questions related to aggressiveness, playing hard-to-get, fear of commitment, decisiveness, etc. Our results came out with average constants to be: rr = .3 and jj = .14, suggesting males have more confidence in their decision-making, and rj = .12 and jr = .26, suggesting females are more attracted to commitment. We plugged those into our MatLab solver and studied the results for various initial feelings of our partners.

With these constants, we set the initial conditions for Juliet at J(0) = 1.0, which denotes she loves Romeo completely. Will this win over any Romeo? The solutions suggest that she can change his mind, but there is a limit to the level of Romeo's dislike for which Juliet can overcome. The limiting point of his initial feelings turned out to be R(0) = -.440, as shown below. Notice how Romeo's moderate dislike is gradually transformed into a happy ending for both of them. However, any more dislike by Romeo dooms the love connection, and Juliet is out of luck.



Finally, let us set the initial conditions for Juliet at her maximum hate level, namely J(0) = -1.0. Does Romeo have a chance for love? Running the approximation program finds that Romeo can win over Juliet with only a moderate love for her, namely at R(0) = .443 (shown below). This suggests that an average boy can win over any average girl, even when she hates him completely at the onset.



According to our model, it seems that a woman can’t always change a man, but a man, with some love, can win over any women’s heart. It turns out she will let him up the balcony for a happy ending.

References:
1. S. H. Strogatz, Love Affairs and Differential Equations, *Mathematics Magazine*, **61** 1988 (35).
2. S. H. Strogatz, *Nonlinear Dynamics and Chaos*, Perseus Books, Cambridge MA, 1994 (138)
3. W.E. Boyce and R.C. DiPrima, *Elementary Differential Equations*, Wiley, 2009 (137-165).
4. *MATLAB Mathematics*, The MathWorks Inc., 1984. See http://www.mathworks.com.