Enclosing roots of polynomial equations and their applications to iterative processes

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Abstract

We introduce a special class of real recurrent polynomials f_m $(m \geq 1)$ of degree m, with unique positive roots s_m, which are decreasing as m increases. The first root s_1, as well as the last one denoted by s_∞ are expressed in closed form, and enclose all s_m $(m > 1)$.

This technique is also used to find weaker than before [5] sufficient convergence conditions for some popular iterative processes converging to solutions of equations.

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1 Introduction

We introduce a special class of recurrent polynomials f_m $(m \geq 1)$ of degree m with real coefficients.

Then, we find sufficient conditions under which each polynomial f_m has a unique positive root s_m, such that $s_{m+1} \leq s_m$ $(m \geq 1)$. The first root s_1, as well as the last one denoted by s_∞ are expressed in simple closed form.

Two applications are provided. In the first one, we show how to use s_1 and s_∞ to locate any s_m belonging in $(s_\infty, s_1]$ $(m \geq 1)$.

In the second one, using this technique on Newton's method (2.19), we show that the famous for its simplicity and clarity Newton–Kantorovich condition (2.43) for solving equations can always replaced by a weaker one (2.22).

Moreover, the ratio of the quadratic convergence of Newton's method 2 q_0 under our approach is smaller than 2 q_K given in [5].

2 Locating roots of polynomials

We need the main result on locating roots of polynomials.

Theorem 2.1 *Let $a > 0$, $b > 0$, and $c < 0$ be given constants. Define polynomials f_m $(m \geq 1)$, g on $[0, +\infty)$ by:*

$$f_m(s) = b\ s^m + a\ s^{m-1} + b\ (s^{m-1} + s^{m-2} + \cdots + 1) + c, \tag{2.1}$$

and

$$g(s) = b\ s^2 + a\ s - a. \tag{2.2}$$

Set

$$d = \frac{2\ a}{a + \sqrt{a^2 + 4\ a\ b}}. \tag{2.3}$$

Assume:

$$d \leq 1 + \frac{b}{c}, \tag{2.4}$$

and

$$a + b + c < 0. \tag{2.5}$$

Then, each polynomial f_m $(m \geq 1)$ has a unique positive root s_m.

Moreover, the following estimates hold for all $m \geq 1$:

$$1 + \frac{b}{c} \leq s^\star \leq s_{m+1} \leq s_m, \tag{2.6}$$

and

$$f_m(s) \leq 0 \quad \text{for all} \quad s \in [0, s_m], \tag{2.7}$$

where,

$$s^\star = \lim_{m \longrightarrow \infty} s_m.$$

Proof.

We shall first show that each polynomial f_m has a unique positive root s_m $(m \geq 1)$.

We have by (2.1), and (2.5):

$$f_1(0) = a + b + c < 0, \tag{2.8}$$

$$f_m(0) = b + c < 0 \qquad (m > 1). \tag{2.9}$$

Moreover for sufficiently large $s > 0$, we also have:

$$f_m(s) > 0 \qquad (m \geq 1). \tag{2.10}$$

It then follows from (2.8)–(2.10), and the the intermediate value theorem that there exists a positive root s_m of polynomial f_m.

Each s_m is the unique positive root of f_m, since

$$f'_m(s) > 0 \;\; (m \geq 1) \qquad (s \geq 0). \tag{2.11}$$

That is the graph of polynomial f_m crosses the positive axis only once.

Furthermore, we shall show estimate (2.6) holds.

We need the relationship between two consecutive polynomials f_m's:

$$\begin{aligned} f_{m+1}(s) &= b\ s^{m+1} + a\ s^m + b\ (s^m + \cdots + 1) + c \\ &= b\ s^m + a\ s^{m-1} + b\ (s^{m-1} + \cdots + 1) + c + \\ &\quad\ a\ s^m - a\ s^{m-1} + b\ s^{m+1} \\ &= f_m(s) + s^{m-1}\ (b\ s^2 + a\ s - a) \\ &= f_m(s) + g(s)\ s^{m-1}. \end{aligned} \tag{2.12}$$

Assume there exists $m \geq 0$, such that

$$s_{m+1} \geq d. \tag{2.13}$$

Note that d is the unique positive root of function g.

We get

$$f_{m+1}(s_{m+1}) = f_m(s_{m+1}) + g(s_{m+1})\ s_{m+1}^{m-1}$$

or

$$f_m(s_{m+1}) \leq 0, \tag{2.14}$$

since $f_{m+1}(s_{m+1}) = 0$, and $g(s_{m+1})\ s_{m+1}^{m-1} \geq 0$, which imply

$$s_{m+1} \leq s_m \qquad (m \geq 1). \tag{2.15}$$

We can certainly choose the last of the s_m's denoted by s_∞, to be s_{m+1}. The point s_∞ has a closed form.

Indeed, polynomial f_m can be written for $s \in [0, 1)$:

$$f_m(s) = b\ s^m + a\ s^{m-1} + b\ \frac{1 - s^{m-2}}{1 - s} + c. \tag{2.16}$$

By letting $m \longrightarrow \infty$, we get:

$$f_\infty(s) = \frac{b}{1-s} + c. \tag{2.17}$$

Function f_∞ has a unique positive root denoted by s_∞, and given by:

$$s_\infty = 1 + \frac{b}{c}. \tag{2.18}$$

Sequence $\{s_m\}$ is non–increasing, bounded below by zero, and as such it converges to $s^\star$ satisfying $s^\star \geq s_\infty$.

Finally, estimate (2.7) certainly holds if $d \leq s_\infty$, which is true by (2.4).

That completes the proof of Theorem 2.1. $\diamondsuit$

As a first application, we show how to locate a root of a polynomial f_m $(m \geq 2)$, using, say e.g. s_{m-1}, and s_∞.

Application 2.2 Let $a = b = 1$, $c = -3$, and $m = 2$. We obtain using (2.1)–(2.3), and (2.18):

$$f_1(s) = s - 1, \qquad f_2(s) = s^2 + 2\ s - 2,$$

$$s_1 = 1, \qquad s_\infty = 0.666666666, \qquad d = .618033989.$$

Conditions (2.4), and (2.5) become:

$$.618033989 < .666666666,$$

and

$$-1 < 0.$$

Hence, the conclusions of Theorem 2.1 hold. In particular, we know $s_2 \in (s_\infty, s_1)$. Actual direct computation justifies the theoretical claim, since

$$s_2 = \sqrt{3} - 1 = .732050808 \in (.666666666,\ 1).$$

Application 2.3 As a second application, we show how to use Theorem 2.1 to derive sufficient convergence conditions for scalar majorizing sequences of certain popular iterative methods such that as Newton's method:

$$x_{n+1} = x_n - F'(x_n)^{-1}\ F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \tag{2.19}$$

where, F is a differentiable operator defined on a convex subset $\mathcal{D}$ of $\mathbb{R}$ with values in $\mathbb{R}$.

It was shown that in [3] (see also [1], [5]) that scalar sequence $\{v_n\}$ $(n \geq 0)$ given by

$$v_0 = 0, \quad v_1 = \eta, \quad v_{n+2} = v_{n+1} + \frac{L\ (v_{n+1} - v_n)^2}{2\ (1 - L_0\ v_{n+1})}, \tag{2.20}$$

where, $\eta > 0$, $L \geq 0$, and $L_0 > 0$ are given constants, is a majorizing sequence for $\{x_n\}$, in the sence that:

$$|x_{n+1} - x_n| \leq v_{n+1} - v_n \qquad (n \geq 0). \tag{2.21}$$

The significance of η, L, L_0 can be found in [3].

Note that the Newton–Kantorovich majorizing sequence is the special case of (2.20) when $L = L_0$. In case $L_0 < L$, our majorizing sequence (2.20) is finer, and under the same computational cost, and hypotheses [1].

Therefore, sufficient convergence conditions for iteration $\{v_n\}$in view of (2.21) lead to the convergence of iteration $\{x_n\}$ to a solution $x^\star$ of equation $F(x) = 0$.

Set $a = L\ \eta$, $b = 2\ L_0\ \eta$, and $c = -2$, in Theorem 2.1.

It is simple algebra to show that conditions (2.4), and (2.5) reduces to (2.22) in the majorizing lemma that follows:

Theorem 2.4 *Assume there exist constants $L_0 \geq 0$, $L \geq 0$, and $\eta \geq 0$, such that:*

$$q_0 = \overline{L}\ \eta \leq \frac{1}{2}, \tag{2.22}$$

where,

$$\overline{L} = \frac{1}{8}\left(L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}\right). \tag{2.23}$$

The inequality in (2.22) is strict if $L_0 = 0$.

Then, sequence $\{t_k\}$ $(k \geq 0)$ given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L_1\ (t_k - t_{k-1})^2}{2\ (1 - L_0\ t_k)} \qquad (k \geq 1), \tag{2.24}$$

is nondecreasing, bounded above by $t^{\star\star}$, and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$L_1 = \begin{cases} L_0 & if \quad k = 1 \\ L & if \quad k > 1 \end{cases},$$

$$t^{\star\star} = \frac{2\ \eta}{2-\delta}, \tag{2.25}$$

$$\delta = \frac{4\ L}{L + \sqrt{L^2 + 8\ L_0\ L}} < 2 \quad \text{for } L_0 \neq 0. \tag{2.26}$$

Moreover the following estimates hold:

$$L_0\ t^\star < 1, \tag{2.27}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}\ (t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1), \tag{2.28}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2\ q_0)^{2^k-1}\ \eta, \quad (k \geq 0), \tag{2.29}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2\ q_0)^{2^k-1}\ \eta}{1-(2\ q_0)^{2^k}}, \quad (2\ q_0 < 1), \quad (k \geq 0). \tag{2.30}$$

Proof. We shall show using induction on k that for all $k \geq 0$:

$$L\ (t_{k+1} - t_k) + \delta\ L_0\ t_{k+1} < \delta, \tag{2.31}$$

$$0 < t_{k+1} - t_k, \tag{2.32}$$

$$L_0\ t_{k+1} < 1, \tag{2.33}$$

and

$$0 < t_{k+2} < t^{\star\star}. \tag{2.34}$$

Estimates (2.31)–(2.33) hold true for $k = 0$ by the initial condition $t_1 = \eta$, and hypothesis (2.22). It then follows from (2.24) that

$$0 < t_2 - t_1 \leq \frac{\delta}{2}\ (t_1 - t_0) \quad \text{and} \quad t_2 \leq \eta + \frac{\delta}{2}\ \eta = \frac{2+\delta}{2}\ \eta < t^{\star\star}.$$

Let us assume estimates (2.31)–(2.34) hold true for all integer values k: $k \leq n+1$ $(n \geq 0)$.

We also get

$$\begin{aligned} t_{k+2} &\leq t_{k+1} + \frac{\delta}{2}\ (t_{k+1} - t_k) \\ &\leq t_k + \frac{\delta}{2}\ (t_k - t_{k-1}) + \frac{\delta}{2}\ (t_{k+1} - t_k) \\ &\leq \eta + \left(\frac{\delta}{2}\right) \eta + \cdots + \left(\frac{\delta}{2}\right)^{k+1} \eta \\ &= \frac{1 - \left(\dfrac{\delta}{2}\right)^{k+2}}{1 - \dfrac{\delta}{2}}\ \eta \\ &< \frac{2\ \eta}{2 - \delta} = t^{\star\star}. \end{aligned} \tag{2.35}$$

We have in turn:

$$L\ (t_{k+2} - t_{k+1}) + \delta\ L_0\ t_{k+2} \leq L\ \left(\frac{\delta}{2}\right)^{k+1} \eta + L_0\ \delta\ \frac{1 - \left(\dfrac{\delta}{2}\right)^{k+2}}{1 - \dfrac{\delta}{2}}\ \eta \leq \delta, \tag{2.36}$$

which shows (2.30)–(2.33) for $k = n + 2$.

Estimate (2.36) can be rewritten as:

$$\left\{ L\ \left(\frac{\delta}{2}\right)^n + \delta\ L_0\ \frac{1 - \left(\dfrac{\delta}{2}\right)^{n+1}}{1 - \dfrac{\delta}{2}} \right\} \eta \leq \delta. \tag{2.37}$$

Estimate (2.37) motivates us to define for $s = \dfrac{\delta}{2}$, the sequence $\{f_n\}$ of polynomials on $[0, +\infty)$ by

$$f_n(s) = \Big(L\ s^{n-1} + 2\ L_0\ (1 + s + s^2 + \cdots + s^n)\Big)\ \eta - 2. \tag{2.38}$$

In view of Theorem 2.1, the induction for (2.31)–(2.34) is completed.

Hence, sequence $\{t_n\}$ is non–decreasing, bounded above by $t^{\star\star}$, and as such that it converges to its unique least upper bound $t^{\star}$. The induction is completed for (2.28).

If $L_0 = 0$, then (2.27) holds trivially. In this case, for $L > 0$, an induction argument shows that

$$t_{k+1} - t_k = \frac{2}{L}\ (2\ q_0)^{2^k} \qquad (k \geq 0),$$

and therefore

$$t_{k+1} = t_1 + (t_2 - t_1) + \cdots + (t_{k+1} - t_k) = \frac{2}{L}\ \sum_{m=0}^{k} (2\ q_0)^m,$$

and

$$t^\star = \lim_{k \to \infty} t_k = \frac{2}{L}\ \sum_{k=0}^{\infty} (2\ q_0)^{2^k}.$$

Clearly, this series converges, since $k \leq 2^k$, $2\ q_0 < 1$, and is bounded above by the number

$$\frac{2}{L}\ \sum_{k=0}^{\infty} (2\ q_0)^k = \frac{4}{L\ (2 - L\ \eta)}.$$

If $L = 0$, then in view of (2.3), $0 \leq L_0 \leq L$, we deduce: $L_0 = 0$, and $t^\star = t_k = \eta$ $(k \geq 1)$.

In the rest of the proof, we assume that $L_0 > 0$.

In order for us to show (2.29) we need the estimate:

$$\frac{1 - \left(\dfrac{\delta}{2}\right)^{k+1}}{1 - \dfrac{\delta}{2}}\ \eta \leq \frac{1}{L_0}\ \left(1 - \left(\frac{\delta}{2}\right)^{k-1}\ \frac{L}{4\ \overline{L}}\right) \qquad (k \geq 1). \tag{2.39}$$

For $k = 1$, (2.39) becomes

$$\left(1 + \frac{\delta}{2}\right)\ \eta \leq \frac{4\ \overline{L} - L}{4\ \overline{L}\ L_0}$$

or

$$\left(1 + \frac{2\ L}{L + \sqrt{L^2 + 8\ L_0\ L}}\right)\ \eta \leq \frac{4\ L_0 - L + \sqrt{L^2 + 8\ L_0\ L}}{L_0\ (4\ L_0 + L + \sqrt{L^2 + 8\ L_0\ L})}$$

In view of (2.22), it suffices to show:

$$\frac{L_0\ (4\ L_0 + L + \sqrt{L^2 + 8\ L_0\ L})\ (3\ L + \sqrt{L^2 + 8\ L_0\ L})}{(L + \sqrt{L^2 + 8\ L_0\ L})\ (4\ L_0 - L + \sqrt{L^2 + 8\ L_0\ L})} \leq 2\ \overline{L},$$

which is true as equality.

Let us now assume estimate (2.39) is true for all integers smaller or equal to k. We must show (2.39) holds for k being $k+1$:

$$\frac{1-\left(\frac{\delta}{2}\right)^{k+2}}{1-\frac{\delta}{2}}\,\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right) \qquad (k \geq 1).$$

or

$$\left(1+\frac{\delta}{2}+\left(\frac{\delta}{2}\right)^{2}+\cdots+\left(\frac{\delta}{2}\right)^{k+1}\right)\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right). \tag{2.40}$$

By the induction hypothesis to show (2.40), it suffices

$$\frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{L}{4\,\overline{L}}\right)+\left(\frac{\delta}{2}\right)^{k+1}\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k}\frac{L}{4\,\overline{L}}\right)$$

or

$$\left(\frac{\delta}{2}\right)^{k+1}\eta \leq \frac{1}{L_0}\left(\left(\frac{\delta}{2}\right)^{k-1}-\left(\frac{\delta}{2}\right)^{k}\right)\frac{L}{4\,\overline{L}}$$

or

$$\delta^2\,\eta \leq \frac{L\,(2-\delta)}{2\,\overline{L}\,L_0}.$$

In view of (2.22) it suffices to show

$$\frac{2\,\overline{L}\,L_0\,\delta^2}{L\,(2-\delta)} \leq 2\,\overline{L},$$

which holds as equality by the choice of δ given by (2.26).

That completes the induction for estimates (2.39).

We shall show (2.29) using induction on $k \geq 0$: Estimate (2.29) is true for $k=0$ by (2.22), (2.24), and (2.26). In order for us to show estimate (2.29) for $k=1$, since $t_2-t_1=\dfrac{L\,(t_1-t_0)^2}{2\,(1-L_0\,t_1)}$, it suffices:

$$\frac{L\,\eta^2}{2\,(1-L_0\,\eta)} \leq \delta\,\overline{L}\,\eta^2$$

or

$$\frac{L}{1-L_0\,\eta} \leq \frac{16\,\overline{L}\,L}{L+\sqrt{L^2+8\,L_0\,L}} \qquad (\eta \neq 0)$$

or

$$\eta \leq \frac{1}{L_0}\left(1 - \frac{L + \sqrt{L^2 + 8\ L_0\ L}}{16\ \overline{L}}\right) \quad (L_0 \neq 0,\ L \neq 0).$$

But by (2.22)

$$\eta \leq \frac{4}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}}.$$

It then suffices to show

$$\frac{4}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}} \leq \frac{1}{L_0}\left(1 - \frac{L + \sqrt{L^2 + 8\ L_0\ L}}{16\ \overline{L}}\right)$$

or

$$\frac{L + \sqrt{L^2 + 8\ L_0\ L}}{16\ \overline{L}} \leq 1 - \frac{4\ L_0}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}}$$

or

$$\frac{L + \sqrt{L^2 + 8\ L_0\ L}}{16\ \overline{L}} \leq \frac{L + \sqrt{L^2 + 8\ L_0\ L}}{L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}}$$

or

$$\overline{L} \geq 0,$$

which is true by (2.23).

Let us assume (2.40) holds for all integers smaller or equal to k. We shall show (2.40) holds for k replaced by $k+1$.

Using (2.24), and the induction hypothesis, we have in turn

$$\begin{aligned} t_{k+2} - t_{k+1} &= \frac{L}{2\ (1 - L_0\ t_{k+1})}\ (t_{k+1} - t_k)^2 \\ &\leq \frac{L}{2\ (1 - L_0\ t_{k+1})}\left(\left(\frac{\delta}{2}\right)^k (2\ q_0)^{2^k - 1}\ \eta\right)^2 \\ &\leq \frac{L}{2\ (1 - L_0\ t_{k+1})}\left(\left(\frac{\delta}{2}\right)^{k-1} (2\ q_0)^{-1}\ \eta\right)\left(\left(\frac{\delta}{2}\right)^{k+1} (2\ q_0)^{2^{k+1} - 1}\ \eta\right) \\ &\leq \left(\frac{\delta}{2}\right)^{k+1} (2\ q_0)^{2^{k+1} - 1}\ \eta, \end{aligned}$$

since,

$$\frac{L}{2\ (1 - L_0\ t_{k+1})}\left(\left(\frac{\delta}{2}\right)^{k-1} (2\ q_0)^{-1}\ \eta\right) \leq 1, \qquad (k \geq 1). \tag{2.41}$$

Indeed, we can show instead of (2.41):

$$t_{k+1} \leq \frac{1}{L_0} \left(1 - \left(\frac{\delta}{2} \right)^{k-1} \frac{L}{4\,\overline{L}} \right),$$

which is true, since by (2.28), and the induction hypothesis:

$$\begin{aligned} t_{k+1} &\leq t_k + \frac{\delta}{2}\,(t_k - t_{k-1}) \\ &\leq t_1 + \frac{\delta}{2}\,(t_1 - t_0) + \cdots + \frac{\delta}{2}\,(t_k - t_{k-1}) \\ &\leq \eta + \left(\frac{\delta}{2}\right) \eta + \cdots + \left(\frac{\delta}{2}\right)^k \eta \\ &= \frac{1 - \left(\frac{\delta}{2}\right)^{k+1}}{1 - \frac{\delta}{2}}\, \eta \\ &\leq \frac{1}{L_0} \left(1 - \left(\frac{\delta}{2} \right)^{k-1} \frac{L}{4\,\overline{L}} \right). \end{aligned}$$

That completes the induction for estimate (2.29).

Using estimate (2.40) for $j \geq k$, we obtain in turn for $2\,q_0 < 1$:

$$\begin{aligned} t_{j+1} - t_k &= (t_{j+1} - t_j) + (t_j - t_{j-1}) + \cdots + (t_{k+1} - t_k) \\ &\leq \left(\left(\frac{\delta}{2}\right)^j (2\,q_0)^{2^j - 1} + \left(\frac{\delta}{2}\right)^{j-1} (2\,q_0)^{2^{j-1}-1} + \cdots + \left(\frac{\delta}{2}\right)^k (2\,q_0)^{2^k - 1} \right) \eta \\ &\leq \left(1 + (2\,q_0)^{2^k} + \left((2\,q_0)^{2^k} \right)^2 + \cdots \right) \left(\frac{\delta}{2}\right)^k (2\,q_0)^{2^k - 1}\, \eta \\ &= \left(\frac{\delta}{2}\right)^k \frac{(2\,q_0)^{2^k - 1}\, \eta}{1 - (2\,q_0)^{2^k}}. \end{aligned} \tag{2.42}$$

Estimate (2.30) follows from (2.42) by letting $j \longrightarrow \infty$.

That completes the proof of Theorem 2.4. $\diamondsuit$

Remark 2.5 In practice

$$L_0 \leq L$$

and $\dfrac{L}{L_0}$ can be arbitrarily large [1]–[4].

Condition (2.22) coincides with the Newton–Kantorovich hypothesis:

$$q_K = L\ \eta \leq \frac{1}{2}, \tag{2.43}$$

if $L = L_0$. Otherwise (2.22) is weaker than (2.43). Moreover the ratio $2\ q_0$ is also smaller than $2\ q_K$.

Hence, (2.22) can replace (2.43) in the Newton–Kantorovich theorem [5].

Example 2.6 Define the scalar function F by $F(x) = c_0\ x + c_1 + c_2\ \sin e^{c_3\ x}$, $x_0 = 0$, where c_i, $i = 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small, $\dfrac{L}{L_0}$ can be arbitrarily large. That is (2.22) may be satisfied but not (2.43).

Example 2.7 Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $U_0 = \{x\ :\ |x - x_0| \leq 1 - \beta\}$, $\beta \in \left[0, \dfrac{1}{2}\right)$, and define function F on U_0 by

$$F(x) = x^3 - \beta. \tag{2.44}$$

Using our hypotheses, we get:

$$\eta = \frac{1}{3}\ (1 - \beta), \quad L_0 = 3 - \beta, \quad \text{and} \quad L = 2\ \ (2 - \beta).$$

The Kantorovich condition (2.43) is violated, since

$$\frac{4}{3}\ (1 - \beta)\ (2 - \beta) > 1 \quad \text{for all} \quad \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method (2.19) converges to $x^\star = \sqrt[3]{\beta}$, starting at $x_0 = 1$.

However, our condition (2.22) is true for all $\beta \in I = \left[.450339002, \dfrac{1}{2}\right)$. Hence, the conclusions of our Theorem 2.4 can apply to solve equation (2.44) for all $\beta \in I$.

Other applications where $L_0 < L$ can be found in [1], [3].

This technique can be used for other Newton–type methods [1]–[7]. Moreover, results obtained here also hold in a Banach space setting.

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