Enclosing roots of polynomial equations and their applications to iterative processes

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Abstract

We introduce a special class of real recurrent polynomials \(f_m\) \((m \geq 1)\) of degree \(m\), with unique positive roots \(s_m\), which are decreasing as \(m\) increases. The first root \(s_1\), as well as the last one denoted by \(s_\infty\) are expressed in closed form, and enclose all \(s_m\) \((m \geq 1)\).

This technique is also used to find weaker than before \([5]\) sufficient convergence conditions for some popular iterative processes converging to solutions of equations.

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1 Introduction

We introduce a special class of recurrent polynomials \(f_m\) \((m \geq 1)\) of degree \(m\) with real coefficients.

Then, we find sufficient conditions under which each polynomial \(f_m\) has a unique positive root \(s_m\), such that \(s_{m+1} \leq s_m\) \((m \geq 1)\). The first root \(s_1\), as well as the last one denoted by \(s_\infty\) are expressed in simple closed form.

Two applications are provided. In the first one, we show how to use \(s_1\) and \(s_\infty\) to locate any \(s_m\) belonging in \((s_\infty, s_1]\) \((m \geq 1)\).

In the second one, using this technique on Newton’s method (2.19), we show that the famous for its simplicity and clarity Newton–Kantorovich condition (2.43) for solving equations can always replaced by a weaker one (2.22).

Moreover, the ratio of the quadratic convergence of Newton’s method \(2 q_0\) under our approach is smaller than \(2 q_K\) given in [5].
2 Locating roots of polynomials

We need the main result on locating roots of polynomials.

**Theorem 2.1** Let $a > 0$, $b > 0$, and $c < 0$ be given constants. Define polynomials $f_m \ (m \geq 1)$, $g$ on $[0, +\infty)$ by:

$$f_m(s) = b \ s^m + a \ s^{m-1} + b \ (s^{m-1} + s^{m-2} + \cdots + 1) + c, \quad (2.1)$$

and

$$g(s) = b \ s^2 + a \ s - a. \quad (2.2)$$

Set

$$d = \frac{2 \ a}{a + \sqrt{a^2 + 4 \ a \ b}}, \quad (2.3)$$

Assume:

$$d \leq 1 + \frac{b}{c}, \quad (2.4)$$

and

$$a + b + c < 0. \quad (2.5)$$

Then, each polynomial $f_m \ (m \geq 1)$ has a unique positive root $s_m$.

Moreover, the following estimates hold for all $m \geq 1$:

$$1 + \frac{b}{c} \leq s^* \leq s_{m+1} \leq s_m, \quad (2.6)$$

and

$$f_m(s) \leq 0 \quad \text{for all} \quad s \in [0, s_m], \quad (2.7)$$

where,

$$s^* = \lim_{m \to \infty} s_m.$$

**Proof.**

We shall first show that each polynomial $f_m$ has a unique positive root $s_m$ \ ($m \geq 1$).

We have by (2.1), and (2.5):

$$f_1(0) = a + b + c < 0, \quad (2.8)$$

$$f_m(0) = b + c < 0 \quad (m > 1). \quad (2.9)$$
Moreover for sufficiently large $s > 0$, we also have:

$$f_m(s) > 0 \quad (m \geq 1). \quad (2.10)$$

It then follows from (2.8)–(2.10), and the intermediate value theorem that there exists a positive root $s_m$ of polynomial $f_m$.

Each $s_m$ is the unique positive root of $f_m$, since

$$f'_m(s) > 0 \quad (m \geq 1) \quad (s \geq 0). \quad (2.11)$$

That is the graph of polynomial $f_m$ crosses the positive axis only once.

Furthermore, we shall show estimate (2.6) holds.

We need the relationship between two consecutive polynomials $f_m$’s:

$$f_{m+1}(s) = b s^{m+1} + a s^m + b (s^m + \cdots + 1) + c$$

$$= b s^m + a s^{m-1} + b (s^{m-1} + \cdots + 1) + c + a s^m - a s^{m-1} + b s^{m+1}$$

$$= f_m(s) + s^{m-1} (b s^2 + a s - a)$$

$$= f_m(s) + g(s) s^{m-1}. \quad (2.12)$$

Assume there exists $m \geq 0$, such that

$$s_{m+1} \geq d. \quad (2.13)$$

Note that $d$ is the unique positive root of function $g$.

We get

$$f_{m+1}(s_{m+1}) = f_m(s_{m+1}) + g(s_{m+1}) s_{m+1}^{m-1}$$

or

$$f_m(s_{m+1}) \leq 0, \quad (2.14)$$

since $f_{m+1}(s_{m+1}) = 0$, and $g(s_{m+1}) s_{m+1}^{m-1} \geq 0$, which imply

$$s_{m+1} \leq s_m \quad (m \geq 1). \quad (2.15)$$

We can certainly choose the last of the $s_m$’s denoted by $s_{\infty}$, to be $s_{m+1}$. The point $s_{\infty}$ has a closed form.

Indeed, polynomial $f_m$ can be written for $s \in [0, 1)$:

$$f_m(s) = b s^m + a s^{m-1} + b \frac{1 - s^{m-2}}{1 - s} + c. \quad (2.16)$$
By letting \( m \to \infty \), we get:

\[
    f_\infty(s) = \frac{b}{1-s} + c. \tag{2.17}
\]

Function \( f_\infty \) has a unique positive root denoted by \( s_\infty \), and given by:

\[
    s_\infty = 1 + \frac{b}{c}. \tag{2.18}
\]

Sequence \( \{s_m\} \) is non-increasing, bounded below by zero, and as such it converges to \( s^* \) satisfying \( s^* \geq s_\infty \).

Finally, estimate (2.7) certainly holds if \( d \leq s_\infty \), which is true by (2.4).

That completes the proof of Theorem 2.1. \( \diamond \)

As a first application, we show how to locate a root of a polynomial \( f_m \) (\( m \geq 2 \)), using, say e.g. \( s_{m-1} \) and \( s_\infty \).

**Application 2.2** Let \( a = b = 1 \), \( c = -3 \), and \( m = 2 \). We obtain using (2.1)–(2.3), and (2.18):

\[
    f_1(s) = s - 1, \quad f_2(s) = s^2 + 2 - s^2 - 2, \\
    s_1 = 1, \quad s_\infty = 0.666666666, \quad d = 0.618033989.
\]

Conditions (2.4), and (2.5) become:

\[
    0.618033989 < 0.666666666,
\]

and

\[
    -1 < 0.
\]

Hence, the conclusions of Theorem 2.1 hold. In particular, we know \( s_2 \in (s_\infty, s_1) \).

Actual direct computation justifies the theoretical claim, since

\[
    s_2 = \sqrt{3} - 1 = 0.732050808 \in (0.666666666, 1).
\]

**Application 2.3** As a second application, we show how to use Theorem 2.1 to derive sufficient convergence conditions for scalar majorizing sequences of certain popular iterative methods such that as Newton’s method:

\[
    x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}), \tag{2.19}
\]

where, \( F \) is a differentiable operator defined on a convex subset \( \mathcal{D} \) of \( \mathbb{R} \) with values in \( \mathbb{R} \).
It was shown that in [3] (see also [1], [5]) that scalar sequence \( \{v_n\} \) \((n \geq 0)\) given by
\[
v_0 = 0, \quad v_1 = \eta, \quad v_{n+2} = v_{n+1} + \frac{L \left(v_{n+1} - v_n\right)^2}{2 \left(1 - L_0 v_{n+1}\right)},
\]
where, \( \eta > 0, \ L \geq 0, \) and \( L_0 > 0 \) are given constants, is a majorizing sequence for \( \{x_n\} \), in the sense that:
\[
|x_{n+1} - x_n| \leq v_{n+1} - v_n \quad (n \geq 0).
\]
(2.21)

The significance of \( \eta, \ L, \ L_0 \) can be found in [3].

Note that the Newton–Kantorovich majorizing sequence is the special case of (2.20) when \( L = L_0 \). In case \( L_0 < L \), our majorizing sequence (2.20) is finer, and under the same computational cost, and hypotheses [1].

Therefore, sufficient convergence conditions for iteration \( \{v_n\} \) in view of (2.21) lead to the convergence of iteration \( \{x_n\} \) to a solution \( x^* \) of equation \( F(x) = 0 \).

Set \( a = L \eta, \ b = 2 L_0 \eta, \) and \( c = -2 \), in Theorem 2.1.

It is simple algebra to show that conditions (2.4), and (2.5) reduces to (2.22) in the majorizing lemma that follows:

**Theorem 2.4** Assume there exist constants \( L_0 \geq 0, \ L \geq 0, \) and \( \eta \geq 0 \), such that:
\[
q_0 = L \eta \leq \frac{1}{2},
\]
(2.22)
where,
\[
\mathcal{L} = \frac{1}{8} \left( L + 4 L_0 + \sqrt{L^2 + 8 L_0 L} \right). \tag{2.23}
\]
The inequality in (2.22) is strict if \( L_0 = 0 \).

Then, sequence \( \{t_k\} \) \((k \geq 0)\) given by
\[
t_0 = 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L_1 \left(t_k - t_{k-1}\right)^2}{2 \left(1 - L_0 t_k\right)} \quad (k \geq 1),
\]
(2.24)
is nondecreasing, bounded above by \( t^{**} \), and converges to its unique least upper bound \( t^* \in [0, t^{**}] \), where
\[
L_1 = \begin{cases} \frac{L_0}{L} & \text{if } k = 1 \\ L & \text{if } k > 1 \end{cases}
\]
\[ t^{**} = \frac{2 \eta}{2 - \delta}, \]  
\[ \delta = \frac{4 L}{L + \sqrt{L^2 + 8 L_0 L}} < 2 \quad \text{for} \quad L_0 \neq 0. \]

Moreover the following estimates hold:

\[ L_0 \ t^* < 1, \]  
\[ 0 \leq t_{k+1} - t_k \leq \frac{\delta}{2} (t_k - t_{k-1}) \leq \cdots \leq \left( \frac{\delta}{2} \right)^k \eta, \quad (k \geq 1), \]  
\[ t_{k+1} - t_k \leq \left( \frac{\delta}{2} \right)^k (2 q_0)^{2k-1} \eta, \quad (k \geq 0), \]  
\[ 0 \leq t^* - t_k \leq \left( \frac{\delta}{2} \right)^k \frac{(2 q_0)^{2k-1} \eta}{1 - (2 q_0)^{2k}}, \quad (2 q_0 < 1), \quad (k \geq 0). \]

**Proof.** We shall show using induction on \( k \) that for all \( k \geq 0 \):

\[ L \ (t_{k+1} - t_k) + \delta \ L_0 \ t_{k+1} < \delta, \]  
\[ 0 < t_{k+1} - t_k, \]  
\[ L_0 \ t_{k+1} < 1, \]  

and

\[ 0 < t_{k+2} < t^{**}. \]

Estimates (2.31)–(2.33) hold true for \( k = 0 \) by the initial condition \( t_1 = \eta \), and hypothesis (2.22). It then follows from (2.24) that

\[ 0 < t_2 - t_1 \leq \frac{\delta}{2} (t_1 - t_0) \quad \text{and} \quad t_2 \leq \eta + \frac{\delta}{2} \eta = \frac{2 + \delta}{2} \eta < t^{**}. \]

Let us assume estimates (2.31)–(2.34) hold true for all integer values \( k \): \( k \leq n + 1 \) (\( n \geq 0 \)).
We also get

\[
\begin{align*}
t_{k+2} & \leq t_{k+1} + \frac{\delta}{2} (t_{k+1} - t_k) \\
& \leq t_k + \frac{\delta}{2} (t_k - t_{k-1}) + \frac{\delta}{2} (t_{k+1} - t_k) \\
& \leq \eta + \left( \frac{\delta}{2} \right)^2 \eta + \cdots + \left( \frac{\delta}{2} \right)^{k+1} \eta \\
& = \frac{1 - \left( \frac{\delta}{2} \right)^{k+2}}{1 - \frac{\delta}{2}} \eta \\
& < \frac{2 \eta}{2 - \delta} = t^{**}.
\end{align*}
\]

We have in turn:

\[
L (t_{k+2} - t_{k+1}) + \delta L_0 t_{k+2} \leq L \left( \frac{\delta}{2} \right)^{k+1} \eta + L_0 \delta \frac{1 - \left( \frac{\delta}{2} \right)^{k+2}}{1 - \frac{\delta}{2}} \eta \leq \delta, \tag{2.36}
\]

which shows (2.30)–(2.33) for \(k = n + 2\).

Estimate (2.36) can be rewritten as:

\[
\left\{ L \left( \frac{\delta}{2} \right)^n + \delta L_0 \frac{1 - \left( \frac{\delta}{2} \right)^{n+1}}{1 - \frac{\delta}{2}} \right\} \eta \leq \delta. \tag{2.37}
\]

Estimate (2.37) motivates us to define for \(s = \frac{\delta}{2}\), the sequence \(\{f_n\}\) of polynomials on \([0, +\infty)\) by

\[
f_n(s) = \left( L s^{n-1} + 2 L_0 (1 + s + s^2 + \cdots + s^n) \right) \eta - 2. \tag{2.38}
\]

In view of Theorem 2.1, the induction for (2.31)–(2.34) is completed.

Hence, sequence \(\{t_n\}\) is non-decreasing, bounded above by \(t^{**}\), and as such that it converges to its unique least upper bound \(t^*\). The induction is completed for (2.28).
If $L_0 = 0$, then (2.27) holds trivially. In this case, for $L > 0$, an induction argument shows that

$$t_{k+1} - t_k = \frac{2}{L} (2q_0)^{2k} \quad (k \geq 0),$$

and therefore

$$t_{k+1} = t_1 + (t_2 - t_1) + \cdots + (t_{k+1} - t_k) = \frac{2}{L} \sum_{m=0}^{k} (2q_0)^{m},$$

and

$$t^* = \lim_{k \to \infty} t_k = \frac{2}{L} \sum_{k=0}^{\infty} (2q_0)^{2k}.$$

Clearly, this series converges, since $k \leq 2^k$, $2q_0 < 1$, and is bounded above by the number

$$\frac{2}{L} \sum_{k=0}^{\infty} (2q_0)^{k} = \frac{4}{L (2 - L \eta)}.$$

If $L = 0$, then in view of (2.3), $0 \leq L_0 \leq L$, we deduce: $L_0 = 0$, and $t^* = t_k = \eta$ ($k \geq 1$).

In the rest of the proof, we assume that $L_0 > 0$.

In order for us to show (2.29) we need the estimate:

$$1 - \left(\frac{\delta}{2}\right)^{k+1} \eta \leq \frac{1}{L_0} \left(1 - \left(\frac{\delta}{2}\right)^{k-1} \frac{L}{4L} \right) \quad (k \geq 1). \quad (2.39)$$

For $k = 1$, (2.39) becomes

$$\left(1 + \frac{\delta}{2}\right) \eta \leq \frac{4L - L}{4L L_0}$$

or

$$\left(1 + \frac{2L}{L + \sqrt{L^2 + 8L_0L}}\right) \eta \leq \frac{4L_0 - L + \sqrt{L^2 + 8L_0L}}{L_0 (4L_0 + L + \sqrt{L^2 + 8L_0L})} \frac{L_0 (4L_0 + L + \sqrt{L^2 + 8L_0L})}{L_0 (4L_0 + L + \sqrt{L^2 + 8L_0L})} \leq 2L,$$

In view of (2.22), it suffices to show:

$$\frac{L_0 (4L_0 + L + \sqrt{L^2 + 8L_0L})}{(L + \sqrt{L^2 + 8L_0L}) (4L_0 - L + \sqrt{L^2 + 8L_0L})} \leq 2L.$$
which is true as equality.

Let us now assume estimate (2.39) is true for all integers smaller or equal to \( k \). We must show (2.39) holds for \( k \) being \( k + 1 \):

\[
\frac{1 - \left( \frac{\delta}{2} \right)^{k+2}}{1 - \frac{\delta}{2}} \eta \leq \frac{1}{L_0} \left( 1 - \left( \frac{\delta}{2} \right)^{k} \frac{L}{4 \mathcal{L}} \right) \quad (k \geq 1).
\]

or

\[
\left( 1 + \frac{\delta}{2} + \left( \frac{\delta}{2} \right)^2 + \cdots + \left( \frac{\delta}{2} \right)^{k+1} \right) \eta \leq \frac{1}{L_0} \left( 1 - \left( \frac{\delta}{2} \right)^{k} \frac{L}{4 \mathcal{L}} \right).
\]  

(2.40)

By the induction hypothesis to show (2.40), it suffices

\[
\frac{1}{L_0} \left( 1 - \left( \frac{\delta}{2} \right)^{k-1} \frac{L}{4 \mathcal{L}} \right) + \left( \frac{\delta}{2} \right)^{k+1} \eta \leq \frac{1}{L_0} \left( 1 - \left( \frac{\delta}{2} \right)^{k} \frac{L}{4 \mathcal{L}} \right)
\]

or

\[
\left( \frac{\delta}{2} \right)^{k+1} \eta \leq \frac{1}{L_0} \left( \left( \frac{\delta}{2} \right)^{k-1} - \left( \frac{\delta}{2} \right)^{k} \right) \frac{L}{4 \mathcal{L}}
\]

or

\[
\delta^2 \eta \leq \frac{L (2 - \delta)}{2 \mathcal{L} L_0}.
\]

In view of (2.22) it suffices to show

\[
\frac{2 \mathcal{L} L_0 \delta^2}{L (2 - \delta)} \leq 2 \mathcal{L},
\]

which holds as equality by the choice of \( \delta \) given by (2.26).

That completes the induction for estimates (2.39).

We shall show (2.29) using induction on \( k \geq 0 \): Estimate (2.29) is true for \( k = 0 \) by (2.22), (2.24), and (2.26). In order for us to show estimate (2.29) for \( k = 1 \), since

\[
t_2 - t_1 = \frac{L}{2} \left( \frac{t_1 - t_0}{t_1} \right)^2,
\]

it suffices:

\[
\frac{L \eta^2}{2 (1 - L_0 \eta)} \leq \delta \mathcal{L} \eta^2
\]

or

\[
\frac{L}{1 - L_0 \eta} \leq \frac{16 \mathcal{L} L}{L + \sqrt{\mathcal{L}^2 + 8 L_0 L}} \quad (\eta \neq 0)
\]
\[ \eta \leq \frac{1}{L_0} \left( 1 - \frac{L + \sqrt{L^2 + 8 L_0 L}}{16 L} \right) \quad (L_0 \neq 0, \ L \neq 0). \]

But by (2.22)
\[ \eta \leq \frac{4}{L + 4 L_0 + \sqrt{L^2 + 8 L_0 L}}. \]

It then suffices to show
\[ \frac{4}{L + 4 L_0 + \sqrt{L^2 + 8 L_0 L}} \leq \frac{1}{L_0} \left( 1 - \frac{L + \sqrt{L^2 + 8 L_0 L}}{16 L} \right) \]

or
\[ \frac{L + \sqrt{L^2 + 8 L_0 L}}{16 L} \leq 1 - \frac{4 L_0}{L + 4 L_0 + \sqrt{L^2 + 8 L_0 L}} \]

or
\[ \frac{L + \sqrt{L^2 + 8 L_0 L}}{16 L} \leq \frac{L + \sqrt{L^2 + 8 L_0 L}}{L + 4 L_0 + \sqrt{L^2 + 8 L_0 L}} \]

or
\[ L \geq 0, \]

which is true by (2.23).

Let us assume (2.40) holds for all integers smaller or equal to \( k \). We shall show (2.40) holds for \( k \) replaced by \( k + 1 \).

Using (2.24), and the induction hypothesis, we have in turn
\[ t_{k+2} - t_{k+1} = \frac{L}{2 (1 - L_0 t_{k+1})} (t_{k+1} - t_k)^2 \]

\[ \leq \frac{L}{2 (1 - L_0 t_{k+1})} \left( \left( \frac{\delta}{2} \right)^k (2 q_0)^{2^k - 1} \eta \right)^2 \]

\[ \leq \frac{L}{2 (1 - L_0 t_{k+1})} \left( \left( \frac{\delta}{2} \right)^{k-1} (2 q_0)^{-1} \eta \right) \left( \left( \frac{\delta}{2} \right)^{k+1} (2 q_0)^{2^{k+1} - 1} \eta \right) \]

\[ \leq \left( \frac{\delta}{2} \right)^{k+1} (2 q_0)^{2^{k+1} - 1} \eta, \]

since,
\[ \frac{L}{2 (1 - L_0 t_{k+1})} \left( \left( \frac{\delta}{2} \right)^{k-1} (2 q_0)^{-1} \eta \right) \leq 1, \quad (k \geq 1). \quad (2.41) \]
Indeed, we can show instead of (2.41):

\[ t_{k+1} \leq \frac{1}{L_0} \left( 1 - \left( \frac{\delta}{2} \right)^{k-1} \frac{L}{4L} \right), \]

which is true, since by (2.28), and the induction hypothesis:

\[
\begin{align*}
t_{k+1} & \leq t_k + \frac{\delta}{2} (t_k - t_{k-1}) \\
& \leq t_1 + \frac{\delta}{2} (t_1 - t_0) + \cdots + \frac{\delta}{2} (t_k - t_{k-1}) \\
& \leq \eta + \left( \frac{\delta}{2} \right) \eta + \cdots + \left( \frac{\delta}{2} \right)^k \eta \\
& = \frac{1 - \left( \frac{\delta}{2} \right)^{k+1}}{1 - \frac{\delta}{2}} \eta \\
& \leq \frac{1}{L_0} \left( 1 - \left( \frac{\delta}{2} \right)^{k-1} \frac{L}{4L} \right). 
\end{align*}
\]

That completes the induction for estimate (2.29).

Using estimate (2.40) for \( j \geq k \), we obtain in turn for \( 2q_0 < 1 \):

\[
\begin{align*}
t_{j+1} - t_k &= (t_{j+1} - t_j) + (t_j - t_{j-1}) + \cdots + (t_{k+1} - t_k) \\
& \leq \left( \left( \frac{\delta}{2} \right)^j (2q_0)^{2^j-1} + \left( \frac{\delta}{2} \right)^{j-1} (2q_0)^{2^{j-1}-1} + \cdots + \left( \frac{\delta}{2} \right)^k (2q_0)^{2^k-1} \right) \eta \\
& \leq \left( 1 + (2q_0)^{2k} + \left( (2q_0)^{2k} \right)^2 + \cdots \right) \left( \frac{\delta}{2} \right)^k (2q_0)^{2k-1} \eta \\
& = \left( \frac{\delta}{2} \right)^k (2q_0)^{2^k-1} \eta \\
& \leq \frac{1}{L_0} \left( 1 - \frac{\delta}{2} \right)^{k-1} L \eta. 
\end{align*}
\]

Estimate (2.30) follows from (2.42) by letting \( j \to \infty \).

That completes the proof of Theorem 2.4.

\[ \diamond \]

**Remark 2.5** In practice

\[ L_0 \leq L \]
and $\frac{L}{L_0}$ can be arbitrarily large [1]–[4].

Condition (2.22) coincides with the Newton–Kantorovich hypothesis:

$$q_K = L \eta \leq \frac{1}{2},$$  \hspace{1cm} (2.43)

if $L = L_0$. Otherwise (2.22) is weaker than (2.43). Moreover the ratio $2 q_0$ is also smaller than $2 q_K$.

Hence, (2.22) can replace (2.43) in the Newton–Kantorovich theorem [5].

**Example 2.6** Define the scalar function $F$ by $F(x) = c_0 x^i + c_1 + c_2 \sin e^{c_3} x$, $x_0 = 0$, where $c_i$, $i = 1, 2, 3$ are given parameters. Then it can easily be seen that for $c_3$ large and $c_2$ sufficiently small, $\frac{L}{L_0}$ can be arbitrarily large. That is (2.22) may be satisfied but not (2.43).

**Example 2.7** Let $X = Y = \mathbb{R}$, $x_0 = 1$, $U_0 = \{x : |x - x_0| \leq 1 - \beta\}$, $\beta \in [0, \frac{1}{2})$, and define function $F$ on $U_0$ by

$$F(x) = x^3 - \beta.$$  \hspace{1cm} (2.44)

Using our hypotheses, we get:

$$\eta = \frac{1}{3} (1 - \beta), \quad L_0 = 3 - \beta, \quad \text{and} \quad L = 2 (2 - \beta).$$

The Kantorovich condition (2.43) is violated, since

$$\frac{4}{3} (1 - \beta) (2 - \beta) > 1 \quad \text{for all} \quad \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton’s method (2.19) converges to $x^* = \sqrt[3]{\beta}$, starting at $x_0 = 1$.

However, our condition (2.22) is true for all $\beta \in I = \left[0.450339002, \frac{1}{2}\right)$. Hence, the conclusions of our Theorem 2.4 can apply to solve equation (2.44) for all $\beta \in I$.

Other applications where $L_0 < L$ can be found in [1], [3].

This technique can be used for other Newton–type methods [1]–[7]. Moreover, results obtained here also hold in a Banach space setting.
References


