EQUILIBRIUM POINTS STABILITY ANALYSIS

OF

COUPLED TWO-SPECIES MODEL

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Abstract: This paper studies inter-competition for resources between two species, and intra-competition for resources among each of the two species. This phenomenon is modeled by a coupled system of two differential equations, of which we find all points of equilibrium. Two of the equilibrium points represent co-existence and co-extinction of the species, respectively, while the other two equilibrium points represent survival of one species and extinction of the other. Local stability conditions for each of the equilibrium points are determined. Next we study each of the four equilibrium points for stability. That is, we determine mathematical conditions on the model parameters under which the scenarios described by the equilibrium points will occur. Such a study of systems of differential equations lays the foundation for the more advanced theories of Chaos in nonlinear dynamical systems.

Key Words: Chaos, Co-existence, Co-extinction, Competition, Equilibrium, Stability.

Introduction

The following model describes competitive interaction between and among two species:

$$\begin{cases} \frac{dX}{dt} = a_1 X - b_1 X^2 - c_1 XY = f(X,Y) \\ \frac{dY}{dt} = a_2 Y - b_2 Y^2 - c_2 XY = g(X,Y) \end{cases} \quad (1)$$

where $X(t)$ is the number of species X at time t, while, $Y(t)$ is the number of species Y at time t. The parameters of this system are described as follows:

a_1 = Net birthrate of species X

b_1 = Mortality rate of species X due to intra-species competition

c_1 = Mortality rate of species X due to inter-species competition with Y

a_2 = Net birthrate of species Y

b_2 = Mortality rate of species Y due to intra-species competition

$c_2 =$ Mortality rate of species Y due to inter-species competition with X

Stability Analysis of Equilibrium Points

To find points of equilibrium for Equation (1), solve the system

$$\begin{cases} f(X,Y)=0 \\ g(X,Y)=0 \end{cases}$$

simultaneously for X and Y to obtain the following four equilibrium points:

1. $\left(\frac{c_1a_2-a_1b_2}{c_1c_2-b_1b_2},\frac{c_2a_1-a_2b_1}{c_1c_2-b_1b_2}\right)$
2. $(0,0)$
3. $\left(\frac{a_1}{b_1},0\right)$
4. $\left(0,\frac{a_2}{b_2}\right)$

We aim to find conditions on the model parameters a_1,b_1,c_1,a_2,b_2,c_2 that lead to stability for each of the above four equilibrium points.

First, start with the co-existence point $\left(\frac{c_1a_2-a_1b_2}{c_1c_2-b_1b_2},\frac{c_2a_1-a_2b_1}{c_1c_2-b_1b_2}\right)$, and use techniques described in [1] to shift it to $(0,0)$ as follows:

Let

$$\alpha=\frac{c_1a_2-a_1b_2}{c_1c_2-b_1b_2}$$

and

$$\beta=\frac{c_2a_1-a_2b_1}{c_1c_2-b_1b_2}.$$

To shift (α,β) to $(0,0)$ let,

$$\begin{aligned} u&=X-\alpha \\ v&=Y-\beta \end{aligned}$$

which yields,

$$X = u + \alpha$$
$$Y = v + \beta$$

so that,

$$X' = u'$$
$$Y' = v'.$$

Substituting into Equation (1) yields

$$u' = a_1(u+\alpha) - b_1(u+\alpha)^2 - c_1(u+\alpha)(v+\beta) = F(u,v)$$
$$v' = a_2(v+\beta) - b_2(v+\beta)^2 - c_2(u+\alpha)(v+\beta) = G(u,v).$$

Linearize $F(u,v)$ and $G(u,v)$ about $(0,0)$ which leads to Jacobian

$$J_{(0,0)} = \begin{bmatrix} F_u(0,0) & F_v(0,0) \\ G_u(0,0) & G_v(0,0) \end{bmatrix}.$$

Specifically,

$$J_{(0,0)} = \begin{bmatrix} a_1 - 2b_1\alpha - c_1\beta & -c_1\alpha \\ -c_2\beta & a_2 - 2b_2\beta - c_2\alpha \end{bmatrix}.$$

To find eigenvalues of the above Jacobian, solve

$$\det\left(\left[J_{(0,0)}\right] - \lambda I\right) = 0$$

which leads to the following quadratic equation:

$$\lambda^2 + (-a_1 + 2b_1\alpha - a_2 + 2b_2\beta + c_2\alpha + c_1\beta)\lambda + \left(a_1a_2 - 2b_2\beta a_1 - c_2\alpha a_1 - 2b_1\alpha a_2 + 4b_1\alpha b_2\beta + 2b_1c_2\alpha^2 - c_1\beta a_2 + 2c_1b_2\beta^2\right) = 0$$

For stability of this equilibrium point the real part of both roots of the above equation must be negative. Theory of quadratic equations leads us to the following conditions therefore required for stability of this equilibrium point:

$$\frac{(a_1b_2 - a_2c_1)(a_1c_2 - a_2b_1)}{(c_1c_2 - b_1b_2)} > 0 \quad \text{and} \quad \frac{b_1(a_2c_1 - a_1b_2) + b_2(a_1c_2 - a_2b_1)}{(c_1c_2 - b_1b_2)} > 0.$$

Next we study stability conditions for co-extinction point $(0,0)$. Using same techniques such as those used for studying stability of the co-existence point, we arrive at following Jacobian:

$$J_{(0,0)} = \begin{bmatrix} F_u(0,0) & F_v(0,0) \\ G_u(0,0) & G_v(0,0) \end{bmatrix}$$

where $F(u,v) = f(X,Y)$ and $G(u,v) = g(X,Y)$, or

$$J_{(0,0)} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

Clearly, the eigenvalues of the above Jacobian are a_1 and a_2, and the required stability conditions are

$$a_1 < 0 \text{ and } a_2 < 0.$$

Third, to test $\left(\frac{a_1}{b_1}, 0\right)$, the point where species X survives and species Y becomes extinct, we first shift it to $(0,0)$. For this purpose let,

$$q = X - \frac{a_1}{b_1}$$
$$r = y - 0$$

Solving for X and Y yields,

$$X = q + \frac{a_1}{b_1}$$
$$Y = r$$

so that,

$$X' = q'$$
$$Y' = r'.$$

Substituting into Equation (1) yields,

$$q' = a_1\left(q + \frac{a_1}{b_1}\right) - b_1\left(q + \frac{a_1}{b_1}\right)^2 - c_1\left(q + \frac{a_1}{b_1}\right)r = F(q,r)$$
$$r' = a_2 r - b_2 r^2 - c_2\left(q + \frac{a_1}{b_1}\right)r = G(q,r) \qquad .$$

Linearize about $(0,0)$ to arrive at the Jacobian:

$$J_{(0,0)}=\begin{bmatrix} F_g(0,0) & F_r(0,0) \\ G_g(0,0) & G_r(0,0) \end{bmatrix}$$

or,

$$J_{(0,0)}=\begin{bmatrix} a_1-2b_1\left(\frac{a_1}{b_1}\right) & -c_1\left(\frac{a_1}{b_1}\right) \\ 0 & a_2-c_2\left(\frac{a_1}{b_1}\right) \end{bmatrix}$$

whose eigenvalues are $-a_1$ and $a_2-c_2\left(\frac{a_1}{b_1}\right)$ which lead to stability conditions,

$$-a_1<0 \text{ and } a_2<\frac{a_1c_2}{b_1}.$$

The fourth equilibrium point, $\left(0,\frac{a_2}{b_2}\right)$ describes when species X goes extinct, and species Y exists. Once again, shift this point to $(0,0)$. To do so, let

$$m=X-0$$
$$n=Y-\frac{a_2}{b_2}$$

Solve for X and Y to get,

$$X=m$$
$$Y=n+\frac{a_2}{b_2}$$

so that,

$$m'=X'$$
$$n'=Y'$$

Substituting into Equation (1) yields,

$$m'=a_1m-b_1m^2-c_1m\left(n+\frac{a_2}{b_2}\right)=F(m,n)$$
$$n'=a_2\left(n+\frac{a_2}{b_2}\right)-b_2\left(n+\frac{a_2}{b_2}\right)^2-c_2m\left(n+\frac{a_2}{b_2}\right)=G(m,n).$$

Once again, linearize around $(0,0)$ to get the Jacobian:

$$J_{(0,0)} = \begin{bmatrix} F_m(0,0) & F_n(0,0) \\ G_m(0,0) & G_n(0,0) \end{bmatrix}$$

which yields,

$$J_{(0,0)} = \begin{bmatrix} a_1 - c_1\left(\frac{a_2}{b_2}\right) & 0 \\ -c_1\left(\frac{a_2}{b_2}\right) & a_2 - 2b_1\left(\frac{a_2}{b_2}\right) \end{bmatrix}$$

whose eigenvalues are $-a_2$ and $a_1 - c_1\left(\frac{a_2}{b_2}\right)$ leading to stability conditions,

$$-a_2 < 0 \text{ and } a_1 < \frac{a_2 c_1}{b_2}.$$

Conclusion and Future Research

In this paper we find local stability conditions for each of the four equilibrium points of a coupled system of two ordinary differential equations (ODEs) that describes inter- and intra-species competition between two species. It can be stated that in a real-life situation, co-existence of species would be most desirable and hence conditions on model parameters could be enforced in order to obtain the same. In a future research project, similar studies could be performed on a coupled system of ODEs that model similar interaction among three or more species.

Bibliography

Y [1] F. Diacu, An ntroduction to ifferential Equations, Order and Chaos, W. H. Freeman and Company, New York, NY. (2000)