MATHEMATICS AS COMMON SENSE: IMAG-INARY ROOTS OF PARABOLIC QUADRATICS

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A popular curricular treatment of "quadratic functions" badly violates the principle of *imparting conceptual understanding*, because it also violates the principles of *mathematical logic*. Its authors erroneously explain how to execute the impossible – namely, to calculate complex-conjugate roots of quadratic functions whose graphs are parabolas within the real-numbers plane.

American mathematics instructors who have been reared through that fallacy commonly accept and perpetuate it – without questioning its mathematical validity or its effects on the students. The intent behind this paper is to empower instructors and authors to provide more correct, and healthier treatments of that topic.

Contrary to commonplace beliefs, no "parabola" quadratic functions can have non-real roots – even though students routinely learn to fake their way through that topic. That mathematical blunder illustrates why students become alienated toward mathematics, and how they become misled into believing that mathematics is permeated with mysterious and irrational rules. If there were laws for protecting students' personal educational health in mathematics, textbooks guilty of so treating the topic of quadratics would be banned from public use – and instructors who insist on using such books would be fined, or worse.

The textbook error stems from authors confusing two classes of functions: the degree-2 polynomials in a single **real** variable, versus the real-coefficient degree-2 polynomials in a single **complex** variable. One is a class of parabolas within the real-numbers plane. The other is a class of non-conformal mappings of the entire plane, onto itself. The error occurs when textbook authors try to use "***i***" for purposes of solving unsolvable equations of the parabolas.

Rescuing the students: In the real world of class courses in mathematics, many instructors find themselves and their students using textbooks which present non-real solutions for parabolic quadratics. Below is shown why that is a mathematical blunder, and how it can be mathematically avoided, circumvented, or repaired. The mathematics, itself, reveals that, and why, the fallacious presentations cannot possibly provide conceptual understanding of the topic. The psychomathematics of how such non-sense instruction undermines students' personal mathematical potentials is tangential to the mathematics, and is not discussed, herein.

Herein, that curricular error with "quadratic functions" is aired from a strictly mathematical viewpoint. Perhaps when the mathematics, itself, is clearly understood by its teachers, the way will be opened for improving curricular treatments and media presentations of that topic. The critique given below can mathematically arm creative instructors to circumvent those fallacious sections of such textbooks, and to provide their students with alternative interpretations which are more correct and more digestible.

The instructor who would rescue the students from the erroneous presentation has some obvious curricular options for doing so. Most of those have the disadvantage of persuading students that the textbook author is worse than a poor writer – perhaps even a mathematical fraud. But the underlying mathematics, itself, offers a more revealing, tolerant, and kindly alternative – of attending "image solutions" for parabolic quadratics.

It might be unrealistic or too extreme to immediately abandon the textbook, especially if is a good one in other regards. Sustaining the book, but skipping that erroneous part of the quadratic theory, fails to enlighten the student about the actual mathematics. Of course, the instructor may "replace" those pages of the text with a correct treatise in the form of a handout or quotation from some other author. But the simplest way to rescue the students is simply to supplement that text with a brief tour through "imag-inary solutions", as sketched, below. [A PowerPoint sketch of what follows may be obtained through e-mail request to the above e-address.]

Quadratics: Above, the phrase, "quadratic functions" is set in quotes – because authors' inconsistencies about its meanings are very much at the heart of the malady. While one popular meaning of that phrase consists of the degree-2 polynomials in a single **real** variable, the error comes from tacitly invoking, within the same context, the real-coefficient degree-2 polynomials in a single **complex** variable, and from confusing those two classes of functions.

Mathematically, that synthesis begs for an even more inclusive definition of "quadratic functions". A function on any numerical domain which admits powers of its variables might justifiably be called, "***quadratic in***" each variable for which, when all other variables are held constant, the (cross-sectional) sub-function is a degree-2 polynomials in the remaining variable. Within that context, it is natural to apply the phrase "quadratic function" to the special cases of functions which are quadratic in some of their variables, and linear in any others – the conic polynomials being simple examples.

Although it would hardly be appropriate to invoke such generality within the core curriculum, such a general definition would provide mathematical legitimacy for the popular practice of tacitly including – under the heading of "quadratic functions" – the real-coefficient degree-2 polynomials in a single **complex** variable.

The Parabolic functions: Authors who define "quadratic functions" to mean degree-2 polynomials in one real variable must sooner or later assert that all such polynomials are vertical-axis parabolas – and also that all such parabolas are degree-2 polynomials. In other words, their chapters which carry the mysterious label of "quadratic functions" actually are intended to be about the parabolic functions.

Oddly, the now traditional curriculum (which calls functions "linear" if their graphs are straight lines) assigns the title, "quadratic", to functions whose graphs are parabolas – functions which have no obvious connection with "quads". Indeed, the malady on which this paper focuses might soon be vanquished merely by changing the chapter heading from "quadratic functions" to "parabolic functions" – since that might suppress authors' attempts to invoke complex numbers.

Just why that change has not yet occurred is a sociological mystery. The superficial reason – that of giving deference to the classical vocabulary of mathematics – is not the full story. Rather, the widely used classical definition of a parabola – in terms of its focus and directrix – is simply too far removed from the central flow of college algebra. Fortunately for students, there exists an equivalent, alternative definition which is much more realistic for novices.

Consider a non-trivial spherical ball (called a "bola"), and a straight line tangent to it. Then bend that line into a curve around the bola in such a way that the resulting curve is coplaner with a great circle of the bola – with each curve-point being equidistant from that circle and from the tangential straight line. Such a curve: (1) lies beside ("para-") that bola – and is called a ***para-bola***; (2) goes through the point of tangency – called the ***vertex*** of the parabola; (3) lies in the plane which contains the tangent and the bola's center – called the parabola's ***center of curvature***; (4) bends around that great circle – called the parabola's ***circle of curvature***; and (5) is symmetric about the curvature-circle's axis though the point of tangency – which also is the ***axis of the parabola***.

When a 2-coordinate square grid is imposed so that the parabola's axis is "vertical" on that grid, the tangent line is necessarily "horizontal" and is a konstant function, k. Since the circle of curvature is either above or below that konstant, the parabola is either c***upped-up*** or ***cupped-down*** from its vertex, in accord with the sign of the radius of curvature. The vertex, itself, is a planer point, (h,k) – where h is the horizontal "longitude" of the parabola's axis.

It is an easy exercise to use the equidistance condition to derive that, for curvature radius, r, and for variable points (x,y) on the parabola, the altitudes, y, at longitudes, x, always are $(1/4r)(x-h)^2+k$. So, every such vertical-axis parabola is a function – and carries a ***parabolic formula*** as $(1/4r)(x-h)^2+k$. Conversely, every function which carries such an $a(x-h)^2+k$ formula is a parabola, whose radius is 1/4a. A little exploration quickly reveals that the multiplier, a, serves as a ***curvature amplitude*** factor for that function – in that at x=h+1 and at x= h-1, the function values are k+a.

It is immediately clear that the domain of function entries of such a ***parabolic function*** is the entire real line, and that its range of function values is from k upward, or from k downward, depending on the direction of curvature. Also obvious is how to solve a parabolic function from its parabolic formula. In fact, solving $a(x-h)^2+k$ for a *range-value*, v, is so direct and simple that there is no need to derive, recall, apply a "quadratic formula" in terms of a, h, and k. [x= h±sqrt((v-k)/a). Therein, the requirement that v is within the values-range of the function guarentees that (v-k) and the amplitude, a, have the same sign, andthat (v-k)/a does have a square root.]

Their polynomial formulas: By so using the parabolic functgions as a portal into the arena of "quadratics", some unnecessary complications are nicely circumvented – notably, manipulating equations in order to complete the square for a polynomial.

Of course, every parabolic formula can be expanded into a standard formula for a degree-2 polynomial over the reals. The converse is not quite obvious, but can easily be achieved, as follows.

Each such polynomial, ax^2+bx+c, is a vertical shift (by +c) of its ax^2+bx component polynomial. The latter has roots at x=0 and at x=-b/a. Midway between those roots (at x=-b/2a), the ax^2+bx polynomial has value, $-b^2/4a$ – whereby, the point ($-b/2a$, $-b^2/4a$) lies on the ax^2+bx curve. So the original ax^2+bx+c curve comprises the point, ($-b/2a$, $c-b^2/4a$). Since the number, c, presents the y-intercept of the original ax^2+bx+c curve, the formula, $c-b^2/4a$, speaks to how that curve's extreme differs from that intercept – which helps for understanding and recalling the formula.

There surely exists a family of parabolic functions which have that point as a vertex – their h as -b/2a and their k as $-b^2/4a$. Exactly one of those has a curvature-radius as 1/4a. That one has the parabolic formula, $a(x-h)^2+k = a(x+b/2a)^2+c-b^2/4a$ – which expands to ax^2+bx+c.

So, each ax^2+bx+c degree-2 polynomial over the reals is, indeed a parabolic function. Its axis is at x= -b/2a. Its amplitude of curvature is exactly a. Its radius of curvature is 1/4a. Its horizontal tangent is at $y=c-b^2/4a$. Its vertex is ($-b/2a$, $c-b^2/4a$).

Instructors who are bent on developing the quadratic-roots formula may prefer to equate $c-b^2/4a$ with $-(b^2-4ac)/4a$. But the geometic meaning of the latter alternative is not nearly as obvious as is that of $c-b^2/4a$.

If ax^2+bx+c has roots, those are symmetric about the parabolic axis. Because its parabolic formula is $a(x+b/2a)^2-(b^2-4ac)/4a$, any roots occur at at $x=(-b/2a)\pm(\text{sqrt}(b^2-4ac))/2a$. So – even without completing the square – the familiar "quadratic-roots" formula comes directly from the vertex-height, $-b^2/4a$, of the ax^2+bx component.

Moreover, the $-(b^2-4ac)/4a$ formula does reveal that $a(x+b/2a)^2 -(b^2-4ac)/4a$ is the (factorable!) difference of two squares ... if, and only if, (b^2-4ac) is non -negative. Finding the formulas for the two (parallel) linear factors can be informative – sinc each line is rooted at a root of the parabolic function.

Of course, those who prefer to complete the squares can formally pass from the function's standard polynomial formula, ax^2+bx+c, to the auxiliary formula, $ax^2+bx+c\pm a(b/2a)^2$, in order to more directly get $a(x+b/2a)^2+c-b^2/4a$. Thereby, completing the square is an alternative way of achieving the polynomial formula's equivalent parabolic formula.

Traditionally, the pedagogical shortcoming of the latter approach has been the need to justify the seemingly artificial square-completion of the ax^2+bx component, by using the auxiliary, $(ax^2+bx)\pm a(b/2a)^2$. But the above development discloses that the square-completion terms, $\pm -b^2/4a$, describe how the vertex-height differs from the intercept. Since $-b^2/4a$ is the height of the vertex of ax^2+bx, adding on $b^2/4a$ to complete the square – to get $a(x+b/2a)^2$ – vertically shifts ax^2+bx, so that the vertex moves onto the x-axis – and $-b^2/4a$ takes it back.

Their quads: Each parabolic function is an ***area*** function. Its (h,k) vertex is a fixed corner of a variable rectangle whose opposite corner is the variable point on the parabola. The base of such a rectangle is (x-h); its altitude is a(x-h); and its area is(x-h)•a(x-h) – which is $a(x-h)^2$. The roles of h and k are to position the rectangles on the coordinate plane.

From that viewpoint, it is realistic to regard the parabolic functions as being ***quadra-tics***. Accordingly, the same term applies to all degree-2 polynomials on the real numbers. But that does not mean that the term need apply ONLY to such functions.

Although such quad-areas seem to pertain only to those polynomial curves which are parabolas, those might be parabolic cross sections of multi-variable functions – functions which, in at least one variable, are degree-2 polynomials on real number spaces.

To further broaden the scope of "quadratics" – so as to comprise ***all*** degree-2 polynomials, regardless of the underlying number system – is to stretch the concept beyond its "quad" meaning. For example, the z^2 function on the complex plane – while usually called "z squared" – is not about "quads", at least not in any obvious way. Nonetheless, under the heading of "quadratic functions", many authors invoke (if only tacitly and mistakenly) degree-2 polynomials that have complex domains. So, we here reluctantly acquiesce by including every polynomial which is of degree-2 in at least one variable as being "quadratic" in that variable.

Complex quadratics: The traditional mathematical blunder comes through viewing the parabolic functions as though they were complex quadratics. The [real] complex functions operate in the real-numbers plane. [The "real" complexes are the points of the real-numbers plane. But each linear system of numbers which has an integers subsystem also has its own plane of complex numbers.] On a 4-quadrant numbers-plane , each point becomes a complex number as soon as complex additions and complex multiplications are imposed onto that plane. Complex additions are vector additions. Complex multiplications are combinations of rotations and sums of scaler multiplications. Complex squaring doubles every point's angle from the positive x-axis – while squaring its modulus. Complex square-root halves every point's angle, while changing its modulus to the square root, thereof.

Each real-coefficients complex "quadratic", az^2+bz+c, is a distortion of the entire real-numbers plane. Its "linear" component, +bz+c, is a horizontal translation (by +(c,0)) of the b-amplification of the plane. The polynomial's "quadratic" component, az^2, doubles angles, squares each modulus, and then amplifies the squared modulus by the multiplier, a. When the quadratic and linear components are added to each other, the resulting distortion of the plane does have two roots, and those are conjugates of each other. But such a distortion of the real-numbers plane definitely is NOT a parabolic curve within that plane.

A case in point : The mathematical error and its resolution are best revealed by a concrete example. Consider the parabolic function, $3(x-4)^2+5$. Being up-cupped from its vertex, (4,5), that function has a values-range as v ≥5. So it cannot have solutions for numbers below 5. In particular, the "equation", [$3(x-4)^2+5=0$] equates the $3(x-4)^2+5$ parabola and the (horizontal) 0-konstant – which nowhere equals the parabola.

The confusion comes through presuming that those two functions are somewhere equated. The equal points (i.e. the intersections) would have to be at real numbers, x, where, the $(x-4)^2$ function equals the -5/3 konstant. [But the $(x-4)^2$ parabola is non-negative!] The fatal stroke is supposing that the x-4 line intersects two konstants, ±sqrt(-5/3).

Of course, the erring authors suppose that all is excused by "defining" the "square root of -1", and giving a name ("**i**") to such a notion. Unfortunately for students who take"-1" to mean the -1 of a system of only real numbers, and take "square root" to mean the square-root associated with real-numbers multiplication, there can be no such thing as a "square root of --1". They are right!

By "merely" introducing **i**, such authors have tacitly shifted the context of discourse from that of real-domain quadratic functions, to that of complex-domain quadratics. In the process, they unknowingly have changed the meanings of "1" and of "-1" – and more importantly, they also changed the meaning of "multiplication" and its associated "squaring" and "square-root".

However, if attention is *knowingly* shifted to the complexes – where "-1" means the (-1,0) point of the real-numbers plane – and where "square-root" means halving the angles, and [real-ly] square-rooting the moduli, then – since $(0,1)^2 = (-1,0)$ – to define "**i**" as meaning (0,1) leads to the theorem that "**i** = sqrt(-1)": the complex square-root of the "-1" complex number.

Under such a shift of contexts, the real-domain quadratic equation, $3(x-4)^2+5=0$, becomes the complex-domain quadratic equation, $3(z-4)^2+5=0$ – whose solutions are 4±**i**•sqrt(5/3). Those are two conjugate points on the plane, (4,5/3) and (4,-5/3), and they have nothing whatsoever to do with the parabolic function, $3(x-4)^2+5$.

Mathematically, there is nothing wrong with so shifting the context of discourse, as long as it is done cleanly and without confusing one context with the other. But authors who inject complex numbers into the core-curriculum for the sole purpose of solving parabolic functions are worse than merely asserting the mathematically impossible. They are unknowingly nurturing students' mathematics-learning distress and their alienation toward the subject.

Parabolic images of parabolic functions: What follows is not readily perceived when quadratic functions are introduced through their degree-2 polynomial formulas. But when the same functions are approached, as above, through their parabolic formulas, the following development is quite simplistic.

The up-cupped $3(x-4)^2+5$ parabolic function can be solved for any number ≥ 5, but for no numbers below 5. However, when its curvature-amplitude is negated, the resulting down-cupped parabolic function, $-3(x-4)^2+5$, can be solved for any number ≤ 5, but for no numbers above 5. The two parabolas are geometrically congruent, and they share the same axis and the same vertex.

Each being the other's (symmetric) reflection across the 5 -konstant, each may be regarded as being the (parabolic) ***image*** of the other. In practice, either might be attended as ***the original***, with the other perhaps being attended as the (original's) image.

Their union is a ***double-parabola***, whose upper branch and lower branch each are functions. The union is, if you wish, a double-valued function. In the same sense that ±sqrt(x) refers to both branches of a curve, the double--branch vertical parabola is $\pm 3(x-4)^2+5$.

Of course, the images for parabolic functions also can be got directly from their polynomial formulas, by use of a conversion formula. For $3x^2-24x+53$ as an original, its parabolic formula is $3(x-4)^2+5$. It is informative to keep the k-tangent apart, to so present the polynomial as $3x^2-24x+48+5$. Thereby, since the image of $3(x-4)^2+5$ is $-3(x-4)^2+5$ – the image's polynomial formula is $-3x^2+24x-48+5$. In general, the image of ax^2+bx+c is got by negating everything but the k. But since that requires calculating, and negating, the ($-b^2/4a$) vertex-height from the konstant, it is just as easy to calculate h and k – and get the image from the parabolic formula, by a(x-h)+k → -a(x-h)+k.

Solving parabolic quadratic functions: Although a parabolic function can be solved only for numbers which lie in one direction from its vertex, its double-parabola can be solved for each real number. The afore cited case of [$3(x-4)^2+5=0$] cannot happen. But that of [$\pm 3(x-4)^2+5=0$] does happen. The auxiliary equation, [$\pm(x-4)^2=-5/3$], reveals that solutions come from the lower branch, where [$-(x-4)^2=-5/3$] yields [x= 4±sqrt(-5/-3)].

In this passage from $3(x-4)^2+5$, as an original parabolic function, to its $-3(x-4)^2+5$ parabolic image, the x= 4±sqrt(-5/-3) are solutions from the image, rather than from the original. So, somewhat reluctantly – but for maximal correct compliance with mistaken textbooks – we may regard x= 4±sqrt(-5/-3) as being the ***image-solutions*** of [$3(x-4)^2+5=0$] ... if you prefer, "the imag-inary" solutions of that equation. [The fact that what today are called "complex numbers" once were called "imaginary numbers" prompts one to wonder if perhaps the here proffered notion of image-parabolas might not be a reincarnation of some ancient ancestor!]

In the context of the double-parabolas, the solutions for [$\pm a(x-h)^2+k = v$] always are got from $(x-h)^2 = \pm(v-k)/a$ ... which is possible only for $(x-h)^2 = |(v-k)/a|$. So, the solutions always are x= h±sqrt(| (v-k)/a |) – being "real" solutions (from the original parabola) or "imaginary" solutions (from the image parabola) depending on the sign of (v-k)/a.

This notion of "imaginary solutions" of parabolic functions even can be expressed symbolically. All that is needed is some means of indicating that the (double-parabola) solutions come from the image branch. In order to maximally accomodate textbooks which (incorrectly) invoke complex numbers, one might use "**:i**" to mean that the solution comes from the image parabola. Thus, the "solutions" to [$3(x-4)^2+5=0$] can be expressed as x= 4±sqrt(|-5/3|):**i** – where Insertion of the colon (or some other separator) keeps those real-number solutions of the image-function from being represented as complex numbers.

Editing the textbook: So, if the textbook's section on "quadratic functions" is actually about parabolic functions, the preceding definition of parabolas can be appended – without conflicting with the book – for purposes of directly achieving the parablic functions and their parabolic formulas. Thereby, parabolic images and the double-parabolas (and their formulas) likewise can be achieved and attended without conflicting with the book.

That sets the stage for achieving "real" and "imaginary" solutions for equations involving parabolic functions. All of that information can be translated into polynomial formulas for the parabolic functions.

However, if the text injects "**i**" – regardless of how far it ventures into the complex numbers – the instructor must decide on how best to deal with the author's venture. One possibility is to circumvent that part of the text – another is to elaborate the text material by attending complex numbers and the (real coefficient) quadratics, az^2+bz+c, and their conjugate solutions. NOT a mathematically responsible option is to try to justify non-real solutions for parabolic functions!

Even circumvention requires explaining the textbook's assertion that "**i** is "the square root of -1". That might be done as follows. "The coordinate plane can be turned into a system of numbers, called the complex numbers, where (-1,0) is the "-1" of the new system of numbers, and (0,1) is called "**i**". In that new system, $(0,1)^2$ is (-1,0), which leads to (0,1) being "the square root of -1". Although we COULD look into solving equations about complex numbers, that is not really part of this course. But anyone who wants to know more about complex numbers can read (the instructor's) handout."

However, if the course also carries the objective of imparting knowledge of the system of (real) complex numbers, then that objective is best pursued in a separate context. Of course, the only rational approach to the complex numbers is through imposing complex-number operations onto a coordinate plane. Otherwise, the assertion that the real-number, -1, could have a real-multiplication square root is logically absurd. [A purely axiomatic definition of complex numbers is beyond the scope of the core-curriculum.]

If the complex numbers are well covered prior to the study of parabolic quadratics, one might then interject a venture into complex quadratics and their (conjugate) solutions. But without taking that route, it is mathematically ridiculous to speak of conjugate, non-real solutions of parabolic functions. [x= h±sqrt((v-k)/a)**i**]. Fortunately, that expression can be salvaged by merely inserting a separator, to mean that the "imaginary" solutions come from from the image-branch of the double-parabola – as x= h±sqrt((v-k)/a) **:i.**