FALLING THROUGH THE CENTER OF THE EARTH

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If a particle with mass m is located in space at a point P, then there is a gravitational field that surrounds the particle. It is given by

$$\vec{E} = \frac{Gm}{r^2}\vec{r}$$

The value of $\vec{E}$ depends on the point Q in space at which one want to know the gravitational field strength induced by the particle at P. Now r is the distance between Q and P and $\vec{r}$ is the unit vector that starts at Q and point toward P. G is the gravitational constant of the universe, and $\vec{E}$ is measured in force per unit mass.

The field strength at Q can be experimentally tested by using a test mass m_T. Since the force on the test mass is $\vec{F} = \vec{E}m_T$, the gravitational field strength at Q is $\vec{E} = \frac{\vec{F}}{m_T}$. So the amount of force on the test particle gives the gravitational field strength at the test particle's location.

If we take a sphere of radius R centered at P and let S denote the surface of the sphere, then we can compute the gravitational flux through S that is generated by our particle. It is defined to be $\int_S \vec{E}\cdot\vec{n}\,dA$. At each point Q on S, $\vec{n}$ is chosen to be a unit vector pointing inward that is at right angles to the tangent space of S at Q. Now $\vec{E}$ is the gravitational field strength and points inward toward P and in this example is in the same direction as $\vec{n}$. So $\vec{E}\cdot\vec{n}$ is positive and the integral is therefore positive as well. Notice that $\vec{E}\cdot\vec{n} = \left|\vec{E}\right| |\vec{n}| \cos(0) = \left|\vec{E}\right| = \frac{Gm}{R^2}$. Since $\vec{E}\cdot\vec{n}$ is constant on S, $\int_S \vec{E}\cdot\vec{n}\,dA = \frac{Gm}{R^2}\cdot$ Surface area of S $= \frac{Gm}{R^2}\cdot 4\pi R^2 = 4\pi Gm$. The gravitational field strength depends only on the mass m of the particle and not on the radius of S. What is really neat is that if S is the surface of any sphere that has our particle at P located inside, then $\int_S \vec{E}\cdot\vec{n}\,dA$ is still $4\pi Gm$.

To show this, suppose S and S' are the surfaces of a sphere centered at P and a sphere containing P that is not centered at P. Then there is a surface C of a cube that contains S and S'. If C is oriented outward and S and S' are oriented inward then

$$\int_C \vec{E}\cdot\vec{n}\,dA + \int_S \vec{E}\cdot\vec{n}\,dA = \int_{C\cup S} \vec{E}\cdot\vec{n}\,dA = \int_V \vec{\nabla}\cdot\vec{E}\,dV$$

with V being the volume trapped between the cube C and the sphere S. The last step is an application of the Divergence Theorem. Now V is a kind of complicated region, but $\vec{\nabla}\cdot\vec{E} = 0$ on V. So $\int_V \vec{\nabla}\cdot\vec{E}\,dV = 0$. This tells us that

$$\int_S \vec{E}\cdot\vec{n}\,dA = -\int_C \vec{E}\cdot\vec{n}\,dA = \int_C \vec{E}\cdot\vec{n}\,dA$$

if C's orientation is changed between the second and third integrals to be oriented inward. A similar argument shows that

$$\int_{S'} \vec{E} \cdot \vec{n}\, dA = -\int_{C} \vec{E} \cdot \vec{n}\, dA = \int_{C} \vec{E} \cdot \vec{n}\, dA$$

as well, so

$$\int_{S} \vec{E} \cdot \vec{n}\, dA = \int_{S'} \vec{E} \cdot \vec{n}\, dA$$

In fact, any compact 2-dimensional manifold M, with manifold taken in the differential geometry sense, and with M oriented inward with P inside has the property that

$$\int_{M} \vec{E} \cdot \vec{n}\, dA = 4\pi Gm$$

I leave it to you, dear reader, to show that $\vec{\nabla} \cdot \vec{E} = 0$ everywhere except, of course, at P where $\vec{E}$ and therefore $\vec{\nabla} \cdot \vec{E}$ is undefined.

If M is a compact 2-dimensional manifold and P is outside of M, then

$$\int_{M} \vec{E} \cdot \vec{n}\, dA = \int_{V} \vec{\nabla} \cdot \vec{E}\, dV = 0$$

with V being the volume inside of M. In general, only the mass inside of M contributes to the net gravitational flux through M. In other words,

$$\int_{M} \vec{E} \cdot \vec{n}\, dA = 4\pi GM_0$$

with M_0 being the total mass inside of M.

We can apply this result to a special problem. Suppose a tunnel is built that passes through the center of the earth and connects two antipodal points on the earth. For example, the North pole and the South pole. Suppose further that all the air is removed from the tunnel to create a vacuum. A person jumping into the tunnel, who has hopefully brought along a personal supply of oxygen, would fall through the center of the earth and arrive at the other side. How long will it take this person to fall through the earth. This is essentially a differential equations problem that requires us to use what we have derived so far to set it up.

Let's suppose the earth is centered at $(0, 0, 0)$ and that the start of the tunnel being jumped into is at $(0, 0, R)$. How long will it take our traveler to reach $(0, 0, -R)$. At any given time in the fall our person is located at a point $(0, 0, r)$. If $r > 0$, the value of $\vec{E}$ can be calculated by letting S be the surface of the sphere of radius r and calculating the gravitational flux through it due to the earth. We make the idealized assumption that the mass of the earth is uniformly distributed throughout the earth.

The important thing to realize is that the symmetry of the problem tells us that at each point on S, the magnitude of $\vec{E}$ is the same and also the direction of $\vec{E}$ is toward the center of the earth or is in the opposite direction of this. It actually has to be toward the center because the gravitational flux through S is positive. Notice that

$$\int_{S} \vec{E} \cdot \vec{n}\, dA = \left|\vec{E}\right| \cdot \text{Surface area of S} = \left|\vec{E}\right| \cdot 4\pi r^2$$

On the other hand

$$\int_S \vec{E} \cdot \vec{n}\, dA = 4\pi G \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} M$$

with M being the total mass of the earth. Only that mass inside S pulls the person down. The rest of the mass of the earth forms a shell around the person that pulls them in every direction, but all the vectors pulling at them add to 0 and have no net effect. From above, we know that

$$\int_S \vec{E} \cdot \vec{n}\, dA = \frac{4\pi G M r^3}{R^3}$$

so that

$$\left|\vec{E}\right| \cdot 4\pi r^2 = \frac{4\pi G M r^3}{R^3}$$

which is to say

$$\left|\vec{E}\right| = \frac{GMr}{R^3}$$

If the mass of our falling person is m_p, then the magnitude of the force on him or her is

$$\left|\vec{F}\right| = \frac{GMrm_p}{R^3}$$

and the acceleration on him, since $\vec{F} = m\vec{a}$, is

$$|\vec{a}| = \frac{GMr}{R^3}$$

In fact

$$a = \frac{-GMr}{R^3}$$

if we model this as a motion in one dimension problem. Notice r is just the location of our person on the z-axis. Now r depends on time and it's second derivative is equal to acceleration. So

$$r'' = (\frac{-GM}{R^3})r$$

or

$$r'' + (\frac{GM}{R^3})r = 0$$

The general solution to this is

$$r = c_1 \cos(\sqrt{\frac{GM}{R^3}}t) + c_2 \sin(\sqrt{\frac{GM}{R^3}}t)$$

Since $r(0) = R$ and $r'(0) = 0$ this means that $c_1 = R$ and $c_2 = 0$. So

$$r = R\cos(\sqrt{\frac{GM}{R^3}}t)$$

and $r = R\cos(\sqrt{\frac{GM}{R^3}}t) = 0$ for the first time when $\sqrt{\frac{GM}{R^3}}t = \frac{\pi}{2}$. This happens when $t = \frac{\pi}{2}\sqrt{\frac{R^3}{GM}}$. So the amount of time it take to go all the way through the earth is twice this amount, or is $\pi\sqrt{\frac{R^3}{GM}}$. This works out to about 42 minutes and 10 seconds. The speed the person is traveling as he or she passes through the center of the earth is $\sqrt{\frac{GM}{R}}$ which is about 7,951 meters per second or about 17,786 miles per hour!