A RESTRUCTURED TRIGONOMETRY COURSE

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1 Introduction

College students take Trigonometry for a variety of reasons. Those concentrating in the sciences and engineering typically enroll in such a course to prepare themselves for Calculus and Differential Equations. Other students take Trigonometry to satisfy a university requirement that they show proficiency in College Algebra and one additional mathematics course. Many universities only offer one version of Trigonometry, so students find themselves in a course not necessarily designed for their specific needs. In this paper I propose an arrangement of topics which might better engage a diverse audience of students in a university trigonometry class.

2 Demarcation of Trigonometric Topics

College Trigonometry as we teach it today is primarily comprised of two broad areas. One deals with trigonometric functions on angles of triangles and the other with trigonometric functions on the real line. In addition, trigonometry courses typically include some related topics such as polar coordinates, parametric curves, analytic geometry, or operations on complex numbers. The author is of the opinion that, when arranging these topics for presentation, the two broad areas of trigonometry can and should be separated to reduce confusion.

To capture the students' attention it makes sense to begin the course with triangle trigonometry using degree measures of angles. College students have a working knowledge of the degree system even before they study trigonometry formally. They know what is meant by making a 90 degree turn and typically understand that spinning around once involves traversing 360 degrees. They are thus more or less ready to study trigonometric ratios of sides of a triangle whose angles are measured in degrees. There is really no need to complicate things by mentioning radians or conversions between degrees and radians in this first part of the trigonometry course. Triangle trigonometry can and should be done in its entirety before explaining to students that one radian has been traversed when a central angle of a circle intercepts an arc the same length as the radius of the circle. This non-intuitive definition requires the presentation of several new concepts that are not pertinent to the study of triangles.

3 Triangles

When trigonometric functions are introduced as ratios of lengths of sides of a right triangle, a world of interesting applications present themselves. It is relatively easy for instructors to engage students as these relations are used to solve accesible triangle problems. Many text books currently in use begin with a presentation of the trigonometric functions on angles and then abruptly change their focus to conversions of units for measuring angles, graphs of trigonometric functions on the real line, or trigonometric identities [1], [2], and [3]. My experience is that some students are left baffled or overwhelmed by such an approach. To me it makes sense to proceed directly from right triangle trigonometry to the trigonometry of general triangles. The Laws of Sines and Cosines as well as Heron's Formula can be presented, and many problems of practical interest can be solved. It is also appropriate to introduce the trigonometric decomposition of planar vectors at this point in the course. Interesting results involving resultant forces and velocities can then be studied. The practical nature of this theory has more potential to engage the typical student than other topics in trigonometry that cannot be visualized to such a degree.

The presentation of trigonometric ratios in triangles can be made somewhat more agreeable by downplaying cotangents, secants, and cosecants. These superfluous functions play an all too important role in many trigonometry text books. Though it is prudent to define them, it is counter productive to litter pages of text with arcane identities involving these trigonometric functions. Fortunately, manufacturers of hand held calculators are constrained by the number of buttons on their devices and have chosen to limit themselves to sines, cosines, and tangents. An argument could be made for not presenting tangents either, but they are somewhat useful for avoiding mention of the hypotenuse in equations encountered in right triangle trigonometry. In addition, it is somewhat reassuring to have notation - in the form of an inverse tangent - for the distribution function of a Cauchy random variable [4, pp. 184-185].

As trigonometric functions are introduced, it is good practice to consider 30-60-90 and isosceles right triangles in particular so that students have the opportunity to evaluate these functions on at least some angles with pencil and paper. As the instructor shifts to more practical problems, students must learn how to use trigonometric tables, trigonometric functions on calculators, or trigonometric capabilities in a computer algebra system. In the discussion of trigonometric relations in right triangles, only acute angles are considered. Consequently, right triangle trigonometry should be followed with a section on trigonometric functions evaluated on angles of more than

90 degrees. The best approach is to define the functions on a domain of angles in standard position in the coordinate plane. Appropriate practice problems will allow the student to become proficient in evaluating the sine, cosine, and tangent of angles of all degree measures in the first revolution. There is really no need to treat negative angles or angles of more than 360 degrees here. Periodicity is best discussed in the second part of the course.

After students learn to evaluate the trigonometric functions on obtuse angles, the Laws of Sines and Cosines and their applications can be studied. Some texts present these trigonometric formulas for general triangles and then dwell on "solving" triangles and dealing with the ambiguous side-side-angle case. It could be argued that a trigonometry text should avoid this approach. When architects and engineers use trigonometry, they deal with tangible triangles. They measure certain angles or sides and determine the measures of other angles and sides applying trigonometric laws. Deducing whether a posited triangle with sides and angles of certain measures exists or is unique is a delightful puzzle to solve for some (and should be treated briefly), but many students are put off by such pursuits. When students have to worry as to whether a triangle problem is even solvable, it detracts from the fun they might otherwise have deducing the width of a river or the pitches of the various sections of a busy roof.

4. Trigonometric Functions on the Real Line

When the presentation of triangle trigonometry is complete, it is a good idea to start right in with the trigonometric functions on the real line. One can begin with central angles of circles and subtended arcs to introduce the concept of radians. After the basic trigonometric functions are defined, the instructor can present the details of the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$, and $y = \arctan x$. This is an excellent time in the course to present the periodicity identities. After shifts, compressions, and dilations have been presented, one can introduce linear combinations of trigonometric functions that arise as solutions of differential equations. Trigonometry students typically are not familiar with derivatives, but their instructor can discuss how certain naturally occuring rates of change can be modeled by equations whose solutions intuitively are periodic and smooth. The obvious illustrative example is a spring mass system [5, pp. 217-219]. To derive

$$a \sin(cx) + b \cos(cx) = d \sin(cx + h)$$

the identity for the sine of a sum is needed. To obtain this identity one can start with the traditional derivation of the expansion of $\cos(\alpha - \beta)$, where α and β are central angles of a unit circle. Then the the identities involving complementary angles and the fact that the sine is odd and the cosine is even can be derived and used to obtain $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$. If these identities are introduced at this point in the course, perhaps the student will appreciate their usefulness. It is only by rewriting the linear combination of a sine and a cosine as a single sine function that one is able to deduce the amplitute and phase shift of the modeled wave.

The reduction, double, and half argument identites for the sine and cosine can follow here so that those taking the course in preparation for Calculus will have this important theory at their disposal. In addition, it is good practice to discuss the product identities that prove useful when working with Fourier series [6, p. 294]. Many texts over emphasize identities involving the other four trigonometric functions and techniques for solving quadratic equations in all six of the trigonometric functions. This educator feels that this is a poor use of time. In practice, the types of trigonometric equations that one might need to solve are of the form $\cos(ax) = b$ or $\sin(ax) = b$. For example, a vector analysis of the the angle of inclination α that will allow an athlete to throw a baseball a maximum horizontal distance, involves solving the equation $\sin(2\alpha) = 1$ [7, p. 873].

5. Supplementary Topics

A solid introductory trigonometry class might include in its latter part the polar coordinate sytem. In addition, students might be introduced to parametric equations involving trigonometric functions and their associated planar curves. Knowledge of these topics is essential for students who will eventually study Multidimensional and Vector Calculus. If the instructor deems it important, time could also be spent at the end of the course on rotations of conic sections. This topic is typically treated in Linear Algebra, but could also be presented in a trigonometry course without using matrix notation [8, p. 233]. Finally, the course could be concluded with DeMoivre's Theorem and its applications in the algebra of complex numbers.

6. Concluding Remarks

The author is of the opinion that trigonometry instructors following certain guidelines on the presentation of topics stand an improved chance of captivating a diverse audience of students. At the same time, these guidelines will allow them to prepare their students for further mathematical studies. Instructors should begin the course with degree measures of angles and then treat triangle trigonometry in its entirety. Only then should they discuss radians and trigonometric functions on the real line.

In addition, superfluous treatment of cotangents, secants, and cosecants, should be avoided. The essential identities should be treated at a point in the course where their usefulness is apparent, and less time should be spent on arcane identities and solving equations that are quadratic in trigonometric expressions. An outline of the proposed course is as follows:

Triangle Trigonometry

- Angles and Degree Measure
- Right Triangles and Definitions of Trigonometric Functins on Acute Angles
- Evaluating Trigonometric Functions on Special Angles
- Evaluating Trigonometric Functions on Angles in the First Revolution
- General Triangles and the Laws of Sines and Cosines
- Heron's Formula and other Applications
- Trigonometric Decomposition of Planar Vectors and Resultants of Vector Sums

Trigonometric Functions on the Real Line

- The Definition of a Radian
- The Unit Circle and Graphs of the Sine, Cosine, Tangent, and Arctangent
- Shifts, Contractions, and Dilations of the Trigonometric Graphs
- Linear Combinations of Sines and Cosines as a Sine
- Derivation of the Sine and Cosine of a Sum of Arguments
- Essential Trigonometric Identities Involving Sines and Cosines

Supplementary Trigonometric Topics

- Polar Coordinates
- Parametric Curves Involving Trigonometric Expressions
- Rotations in Analytic Geometry

- DeMoivre's Theorem and the Algebra of Complex Numbers

To be Downplayed

- Cotangents, Secants, and Cosecants
- Arcane Identities
- Solving Quadratic Equations in Trigonometric Expresions

References

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