THE WRONSKIAN AND LINEAR INDEPENDENCE

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In my Differential Equations' class we encountered the following problem in the section discussing the structure of solutions to 2nd order linear differential homogeneous equations.

Let f(t) and g(t) be continuously differentiable real-valued functions on an open interval I. Is it true that the Wronskian of f and g is zero on I iff f and g are linearly dependent on I?

For the sake of completeness, let us recall that a set of vectors $S = \{x_1, x_2, \ldots, x_n\}$ in some vector space V is linearly dependent iff there exist scalars $c_1, c_2, \ldots c_n$, not all 0, such that $c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$. Otherwise, the set S is called linearly independent.
It is easy to show that a set of two vectors $\{x_1, x_2\}$ is linearly dependent if and only if one of the vectors is a multiple of the other; that is, either $x_1 = cx_2$ or $x_2 = cx_1$ for some scalar c. **Note**: a set containing the zero vector is always linearly dependent so c can be 0.

The Wronskian for two real-valued differentiable functions f and g is defined by

$$W\left(f(t),g(t)\right) = \det\begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} = f(t)g'(t) - f'(t)g(t).$$

If f and g are linearly dependent on some interval I, then, WLOG we may assume f = cg for some scalar c, so that the Wronskian becomes

$$W\left(f(t),g(t)\right) = \det\begin{bmatrix} cg(t) & g(t) \\ cg'(t) & g'(t) \end{bmatrix} = cg(t)g'(t) - cg'(t)g(t) = 0$$

on I. In other words, if f and g are linearly dependent on I, the Wronskian $W(f(t),g(t))$ is identically 0 on I.

That the converse is false is shown by the following example.

Let $I = (-\infty,\infty)$, $f(t) = t^3$ and $g(t) = |t|^3 = \begin{cases} t^3, & t \geq 0 \\ -t^3, & t<0 \end{cases}$.

Then, $f'(t) = 3t^2$ and $g'(t) = \begin{cases} 3t^2, & t \geq 0 \\ -3t^2, & t<0 \end{cases}$.

For $t \geq 0$ the two columns of the Wronskian are the same whence the determinant equals 0. For t<0 the second column is the negative of the first column so that, again, the determinant equals 0. Therefore, the Wronskian is identically zero on all of $I = (-\infty, \infty)$. It is clear that the equation $f(t) = t^3 = c|t^3| = cg(t)$ cannot hold for all t in I since $t>0$ implies $c = 1$ and $t<0$ implies $c = -1$. Likewise $g = cf$ is not possible. Hence, f and g are linearly independent on I. Of course, if we choose I to be $(0,\infty)$ or $(-\infty,0)$, then f and g are linearly dependent on the interval I.

The following Theorem occurs in the section on linear homogeneous 2nd order differential equations.

Text Theorem: Let f and g be two solutions of $x'' + a(t)x' + b(t)x = 0$ on an open interval I, where a(t) and b(t) are continuous on I. The Wronskian of f and g is zero at every point of the interval I if and only if f and g are linearly dependent on I.

As shown above the "if" part of the result does not require that the functions be solutions to the differential equation but only on linear dependence. The argument below for the proof of the converse is taken from our text.

Suppose the Wronskian

$$W(f(t),g(t)) = f(t)g'(t) - f'(t)g(t) = 0$$

for every t in I. Then we can write

$$\frac{f'(t)}{f(t)} = \frac{g'(t)}{g(t)}$$

as long as f(t) and g(t) are not zero. Integrating the equation gives

$$\ln|f(t)| = \ln|g(t)| + B,$$

where B is an arbitrary constant of integration. When we exponentiate both sides, we find that

$$f(t) = Cg(t)$$

where $C = \pm e^B$ depending on the signs of f and g. Hence f and g are linearly dependent.

Note that this argument does not make use of the requirement that the functions f and g are solutions of the differential equation but does require that f(t) and g(t) never equal 0 on I. So, the question arises as to the need for this condition when f(t) or g(t) equal 0 on I and there is no discussion in the text about this case. Clearly, our counterexample works because $f(0) = g(0) = 0$ and the "proof" above requires that f(t) and g(t) are not zero. Thus if we restrict our interval I so it does not contain $t = 0$, then f and g in our "counterexample" will be linearly dependent on I.

The purpose of this talk is to explore the question: how do we complete the "proof" of the Theorem from the text when f or g takes the value 0 in I.

Case 1. $f(t_0) = 0$ but $g(t_0) \neq 0$ for some t_0 in (a,b).

Since the Wronskian

$$W(f(t),g(t)) = f(t)g'(t) - f'(t)g(t)$$

is identically zero on I it follows that $f'(t_0) = 0$. But f is a solution to the second order homogeneous linear differential equation $x''(t)+a(t)x'(t)+b(t)x(t) = 0$ with continuous "coefficients" a(t) and b(t) on the open interval I and satisfies the initial conditions $x(t_0) = x'(t_0) = 0$ for some t_0 in I. By the Uniqueness and Existence Theorem for such differential equations, there is one and only one function which can satisfy any given set of initial conditions on I. Clearly the zero function is a solution to the homogeneous differential equation on I and satisfies these initial conditions, so f must, in fact, be the zero function on I. Of course, $\{f(t) \equiv 0, g(t)\}$ is a linearly dependent set on I. Clearly, the same argument would work if $f(t_0) \neq 0$ and $g(t_0) = 0$.

Case 2. $f(t_0) = g(t_0) = 0$ and t_0 is the only point in I = (a,b) for which f(t) or g(t) is 0.

Let $I_1 = (a,t_0)$ and $I_2 = (t_0,b)$. Since f and g do not equal 0 at any point of I_1 or I_2, the argument from the text given above can be used to conclude that $f = c_1 g$ on I_1 and $f = c_2 g$ on I_2. Then, the problem is to show that $c_1 = c_2$. Clearly $f' = c_1 g'$ on I_1 and $f' = c_2 g'$ on I_2. Since f and g are solutions to a second order differential equation, their second derivatives must exist whence it follows that f' and g' are continuous on I and thus on I_1 and I_2. If $g'(t_0) = 0$, then the previous argument for Case 1 shows that g is the zero function and $\{f(t), g(t) \equiv 0\}$ is linearly dependent. Thus, under the condition that $g'(t_0) \neq 0$, we take the left hand limit in I_1 at t_0 to obtain $f'(t_0) = c_1 g'(t_0)$ and the right-hand limit in I_2 at t_0 to obtain $f'(t_0) = c_2 g'(t_0)$. Then, $g'(t_0) \neq 0$ implies that $c_1 = c_2 = c$ so that $\{f = cg, g\}$ is linearly dependent on I. This argument can be repeated "piece-by-piece" to cover Case 3 where f and g are both 0 on a finite subset of the interval I.

Case 3. f and g are both 0 on a finite set of points S in I = (a, b).

Let $S = \{t_1 < t_2 < \ldots < t_n\}$ and define $I_0 = (a, t_1)$, $I_1 = (t_1, t_2)$, …, $I_{n-1} = (t_{n-1}, t_n)$, $I_n = (t_n, b)$.

Then, as in Case 2, for each j = 0,1,…,n, there is a c_j such that $f = c_j g$ on I_j. If $g'(t_{j_0}) = 0$ for some j_0 such that $0 \leq j_0 \leq n$, then it follows that g is the zero function on I. Suppose $g'(t_j) \neq 0$ for all j. Using exactly the same limit argument as in Case 2, we obtain $f(t_j) = c_j g(t_j) = c_{j+1} g(t_j)$ whence it follows that $c_j = c_{j+1}$ for j = 0,1, ...,n-1. Letting c be the common value of c_j we obtain f = cg on I whence $\{f = cg, g\}$ is linearly dependent on I.

Case 4a. There are infinitely many points S in I where f and g are both 0 and this set S has a limit point t_0 in I.

In this case there is a sequence $\{t_n\}$ of points in I that converges to t_0 and $f(t_n) = g(t_n) = 0$. Then, continuity implies that $f(t_0) = g(t_0) = 0$. Moreover, $f'(t_0) = \lim_{n\to\infty} \frac{f(t_n) - f(t_0)}{t_n - t_0} = 0$ so that $f(t_0) = f'(t_0) = 0$. Then, as in Case 1, f must be the zero function and $\{f(t) \equiv 0, g(t)\}$ is linearly dependent on I. Note this argument really only requires that f <u>OR</u> g be 0 on an infinite set of points S in I that has a limit point in I.

Case 4b. f and g are both 0 on an infinite subset S of the interval I = (a,b) but S has no limit point in I. (For example, a limit point of S might be an endpoint of I or I might be unbounded.)

If I is bounded, let L = b - a and define $I_n = (a - L/(n+1), a + L/n)$. If any I_n contained an infinite number of points of S then $S \cap I_n$ would have a limit point in the bounded interval [a + L/n+1, a+1/n] and hence in I contrary to the supposition that S has no limit point in I. Thus, each I_n contains only finitely many points where f and g are both 0. Then the "piece-by-piece" argument used in Case 2 can be used here to to show that f and g are linearly dependent on each I_n. In other words, either g is the zero function on I or for each n there is a constant c_n such that $f = c_n g$ on I_n. The argument used for Case 1 (and as applied to Case 3) in which there was exactly one point t_o where f and g are both 0 can be used to show that $c_n = c_{n+1}$ for any n. If we denote by c the common value of c_n for all n, we obtain f = cg on I. Actually, what we have shown is that for any t in I, there is a constant c, independent of the choice of t, such that f(t) = cg(t) because for any choice of t, there is an I_n such that t belongs to I_n and $f(t) = c_n g(t) = cg(t)$. Thus, {f, g} is linearly dependent on all of I. If I is unbounded then we can partition I into intervals I_n of length one (e.g. if $I = (a, \infty)$ then $I_n = (a+n-1, a+n)$) and repeat the previous argument for a bounded interval to show that f and g are linearly dependent on I.

This now completes the proof of the Text Theorem in all possible cases.