MATHEMATICS AS COMMON SENSE: UPPER AND LOWER LIMITS FOR FUNCTIONS

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Abstract:
The "delta-epsilonic" definition for the limits of functions has proven so elusive for calculus students that many "reformers" have forsaken mathematical rigor, in favor of more "conceptual understanding." But the Dedekind-Frechet ("zoom") concept of limits is both common-sensible to students, and fully rigorous. Thereby, students can learn "lim" as an operator that carries each function into its best continuous approximation — and as a functional that gives numbers to functions having unbounded domains. "Delta-epsilonics" becomes a theorem.

Source:
A major tool of the science of mathematics instructology is the ***syllabus methods*** adaptation of ***critical path methods***, from the managerial sciences. When used in the context of clinical research, syllabus methods uncover many maladies of curricular mathematics syllabi, and ways of improving them. This description of a re-formed syllabus is written to mathematicians who already know the mathematical subject matter, but who are concerned about improving the effectiveness of instruction.
Alternative versions for teachers and students are slated for e-publication on http://www.mathsense.org/secrets.htm.

Circumventing epsilonics:
In the first course in calculus, students' initial difficulties with the notions of "limits of functions" long have impeded effective progress of learning and instruction. Clinical research discloses that students' conceptual difficulties with the topic are not intrinsic to the nature of limits, as such. Rather, those difficulties typically stem from traditional curricular reliance on epsilonics, for purposes of DEFINING the limits of functions.

Of course, that assertion is established by using an alternative mathematical definition which does not entail such conceptual difficulties. This paper is about such an alternative — and about re-forming the definitions of a function's limits, so as to make those concepts more comprehensible when first encountered.

In normal practice, the calculus instructor uses a calculus textbook — and cannot avoid somehow dealing with the epsilon-bands properties of limits of functions. But before doing so, the learning-conscious instructor may supplement the text with a conceptual bridge, in the form of an alternative definition — and thereby later achieve the banding property as a common-sensible theorem.

As presently approached in beginning calculus, the basic role of function-limits is to focus on a point **near to** (and possibly within) the domain of a function, for knowledge about the function-values of nearby domain-points. Such knowledge often is expressed: "as $x \to a$, $f(x) \to L$" — verbally, "as x converges to a, f(x) converges to L."

Epsilonics being an unduly complicated way of defining such dual convergence, most students eventually come to **understand** function-limits, through acquiring the percept, "as $x \to a$, $f(x) \to L$". In fact, the normal process for actually finding the function's limit at a is to examine an x for which $x \to a$, to find the L for which $f(x) \to L$. In a sense, this paper is about developing the condition, "as $x \to a$, $f(x) \to L$", through a rigorous, but easily comprehensible initial definition for limits of functions.

The over-complexity of the epsilonics definition results partly from the traditional rhetoric's reliance on an ***implicit*** definition. The occurrence of epsilon-bands is contingent on possible existence of a qualifying number — "... if there exists a number, L, for which ..., then L is called ...." That definition inversely manifests an "engineering-problem approach" to function-limits: "How can that function be used for generating tolerable results?" — or — "Given tolerance-bands, L ±ε , find domain-restrictions, a ±δ, which will suffice." In contrast, the epsilonics definition of function-limits does not actually reveal how to FIND the needed a ±δ nests. Rather, it begins with a point, a, to verify whether or not a particular a ±δ nest will suffice.

Of special practical importance is that such verification often yields an "epsilon-compliance" formula which can be applied for purposes of calculating adequate restrictions of the function's domain — so revealing a delta, for any chosen epsilon,. Achieving such a formula can be seriously important for some special purposes. But that is no excuse for clinging to epsilonics as means for trying to guide students to achieve their initial conceptual under-standing of limits.

Herein, the "x→a" notion initially is replaced by the concept of an intervals-nest that converges to a number or to an "infinity". In the mode of David Frechet's concepts of the **S**up(remum) and **I**nf(imum) of real-valued functions, a corresponding nest of function-ranges generates a (graphic) function-windows nest which **converges** to an interval, and possibly to a point. On that standpoint, the "x→a" notion later is given a mathematically firm meaning. However, effective instructional use of all of those ingredients depends on students owning some version of the ***continuity condition*** of the real numbers.

A number-line approach to limits:
In a very fundamental sense, the notion of a function's limits at points is a generalization of the process of removing discontinuities of number-lines. "The limit" of a rational-numbers line is a real-numbers line. More specifically, conceptual understanding of the limits of functions is rooted to understanding that some *dense* number-lines have a density of (Dedekind) *holes* — so constituting *infinitesimal sieves*.

Even students of introductory algebra can readily grasp the continuity condition by examining the kinds of Dedekind cuts that occur with various kinds of linearly-ordered systems of numbers. At that level, the most illuminating sieve is the dense line of decimal-numbers (whose simple decimal-point numerals are of finite length). Since the non-decimal fractions are missing, the line of decimal numbers has a density of holes — and an infinitude of **bounded intervals which have no endpoints**.

Such instructive recourse to infinitesimal sieves calls for an "intervals" concept which is far more general than the traditional curricular one. Along the line of real numbers, it suffices to describe intervals in terms of their endpoints. But for the more general case, "inter-" is taken to mean "between". An interval of a line is a between-ness closed subset — for every pair of points, therein, all line-points that are in between those two also lie within that interval.

That concept allows for intervals of lines which have holes — and for no-endpoint intervals which end at holes — and for "infinity" nests of singly-bounded, but endless intervals. Especially pertinent, that concept also allows delta-intervals, each of which spans only a single Dedekind hole.

Of course, each pair of line-points identifies a closed interval. However, the use of (coordinate point) ordered pairs to express intervals can introduce initial confusions which are best avoided. So, for a compromise, intervals which have two extrema may be expressed as [x≤≤y]. Whether or not ALL intervals can be expressed in terms of their "ends" depends on the topology of the number-line.

Beginning calculus students' initial conceptual difficulties with function-limits often stem from their expectations that "limits" should be limits OF something — and the traditional curriculum fails to clarify, "limits of what?" But the needed clarification best begins within a context that is conceptually simpler than the limits of functions — and Dedekind cuts of number-lines suffice.

A cut of a line is simply an (L,U) disjoint pair of non-empty intervals whose union is the entire line. Every non-end line-point, x, defines two distinct cuts — its *lower-cut* having that point as a max of its lower part — (-∞≤≤x] — and its *upper-cut* having the same point as the min of its upper part — [x≤≤+∞). The ***discrete*** lines (e.g. of integers) serve as contra-illustrations which set the stage by contrasting with limit points of the line. Along lines of the discrete kind, every cut is a *gap* — having both a max for its lower part, and a min for its upper part — (-∞≤≤x]&[y≤≤+∞), with x<y and no points between x and y. From that condition the concept of an interval's ***endpoints*** simplistically arises even within the context of discrete lines — as the max or min of an interval. But while endpoints of that kind do serve as *bounds* for intervals, they do not exhibit "the **pressures**" that we want for "limit-points".

In the contrast, a dense-sieve line has 3 kinds of cuts. One kind of cuts consists of ***holes*** — where U has no min, and L has no max. Otherwise, the remaining two kinds of *numbered cuts* have ***cut-numbers***. Each line-point, x, is the cut number both for an upper-cut, (-∞≤≤x)&[x≤≤+∞), in which x is the min of the upper-part — and also for a lower-cut, (-∞≤≤x]&(x≤≤+∞), in which x is the max of the lower-part.

However, the density dictates that x is crowded by other points from each of the those four outer intervals — making x a *lower limit-point* of each upper part, and also an *upper limit-point* of each lower part. But while x is the in-common endpoint of ***each of those four*** unbounded intervals, it is an extremum of only two of them.

Thus, the concept of extrema-***endpoints is*** generalized — to also include non-extrema endpoints that lie outside the interval. The essence of limit-ness simplistically surfaces through the context of interval-ends.

A ***limit-point*** of a subset of a number-line is "infinitesimally" ***near*** to that set, whether or not it also lies within that set. But students most easily grasp that concept through the context of intervals of dense lines. When an interval has an *upper end*, that end is the ***upper limit*** of that interval — and when it has a *lower end*, that end is the ***lower limit*** of that interval.

Of course, if those ends are end***points***, those are the sup and inf of that interval — but it is distractive to introduce additional vocabulary until it serves additional purposes. In the calculus, however, it often is expedient to regard upper and lower "infinities" as "ends" of line-intervals. Indeed, it can be helpful even to regard Dedekind holes as being "ends" of intervals.

Along the line of decimal numbers, every point is a limit-***point*** — being a limit of other decimal numbers. But density dictates that each line-**hole** also is a limit of decimal numbers. Although a hole-cut cannot be endpoint-described, the hole, itself, can be regarded as being an "end" for both parts — and given a name, say h or e or pi. Thereby, as with ±∞ — even though, h is not a number — the hole-cut is (-∞≤≤h)&(h≤≤+∞). So, to the extent that we can generate a vocabulary for the holes (e.g. by using non-terminating decimal-points), all intervals can be described by their ends — even when those ends are not endPOINTS .

On the surface, so attending Dedekind holes might seem to be extraneous for beginning calculus. But the holes provide the context for the "delta intervals" on which even the sup/inf definition of function-limits depends. Indeed, achieving the limit-values for functions often is done by using delta-intervals to generate holes ***in the function,*** and by then inserting limit-points to fill those holes.

En route, our numeric approach to limits takes an interim trek through ***nested intervals***. That serves to advance the concept of "limit" a step beyond limits as ends of intervals — while not yet broaching the complexity of function-limits. But that turn is not actually tangential. The nests also are crucial to the sup/inf definition of function-limits.

A ***nest*** is simply an inclusion-ordered chain of non-empty intervals. The nest of all upper-intervals, and the nest of all lower-intervals, serve as viable definitions of upper infinity and lower infinity. Classical epsilonics uses three special kinds of nests — its epsilon nests, its delta-nests, and its "infinity" nests of outer intervals. But some of the built-in postulates about those nests are not essential — and serve only to inhibit acquisition of the concepts of function-limits. The more general notion of nests is easier to grasp.

For each nest, a non-empty intersection of all of its member-intervals is an interval — which might or might not be a member of the nest. In a sense, each nest "converges" to its intersection — but more precisely, its intervals *concur* there. To speak of a nest "concurring" allows "convergence" to mean, as usual, the specific cases where the nested intervals converge to a single point — or to a single hole or an infinity — as the ***limit of that nest***.

Each decimal number is a *limit point* of a convergent nest of intervals — but some such decimals-line nests converge to holes in that line. To somehow fill those holes (e.g. with non-terminating decimals, or with delta-nests) amounts to achieving limit-points for ALL convergent nests. Scenarios of that kind can serve as a prolog for the passage from functions which have a density of holes (such as the equality function over the rationals), to their continuous extensions.

Nests of function-windows:

The typical calculus book ignores some notions which are fundamental standpoints for under-standing the limits of functions. One of those is the concept of a function's nests of ***function-windows***.

Along a line of real numbers, each interval of the line includes a (possibly empty) interval of the chosen function's domain. In turn, that domain-interval identifies a particular sub-function as an ***interval-restriction*** of the original function. Thereby, each such interval-restriction of the function's domain also yields a corresponding restriction of that function's range. The resulting restricted range lies fully within a minimal encompassing line-interval. The latter, if non-empty, has "end" limits which are real-number endpoints, or are one or both of the "infinities". Thus, through each interval-restriction of its domain, that function ***induces*** a corresponding interval-restriction of its range.

Whenever the restricted domain and range are bounded, the restricted function's configuration is quite naturally linked to the technology of graphics calculators. Since the calculator's "x-interval" comprises its endpoints, it is closed — and it also restricts the function's domain. The restricted function's ***window*** is completed by the corresponding minimal closed "y-interval" which includes the restricted range — which the calculator might or might not be able to accomodate. The y-interval's ends — whether numbers or infinities — are automatically set by the restricted function. Conceptually, that "window frame" builds upon the students' experience with the calculators.

So, for each function, f, every x-nest (of closed intervals) connotes an associated nest of restricted f-domains — and its nest of f-restrictions — and its nest of f-restrictions' ranges — and its **induced nest** of (closed) "y-intervals" — yielding ***a nest of f-windows*** over the concurrence of the x-nest. In the direction of smaller subsets, such a nest of border-including f-windows is what calculators call an "in-zoom". But since the y-nests are induced by f, all zoom-in's on f are controlled by the x-nests. Even through classical epsilonics, students' awareness of such f-nests is crucial to under-standing of function limits.

In the traditional mode, the instructional syllabus would immediately proceed directly to the definition of f-limit at points, p. But a much stronger under-standing is got by first acquiring some standpoints which justify focusing on what is to be defined. Those come from examining what happens to the f-induced y-nest, at the concurrence of the x-nest — especially when the x-nest converges to a line-point or to an "infinity".

It is well worth examining how various selected x-intervals**:** restrict functions, define function-windows, restrict ranges, and induce the windows' y-intervals — to detect what happens when the x-intervals are expanded or shrunk. But the concepts of function-limits are all about how the f-induced y-nest concurs for an x-nest which actually **converges** (to a point/hole/infinity limit-thing). That is the context in which calculus students most readily grasp the nature of function-limits — ***as domf is shrunk toward (... some number or infinity ...), ranf is pressed toward (... some interval or number).***

It is noteworthy that differing x-nests which converge to an in-common limit might yield greatly differing y-nests. But in beginning calculus, only a handful of x-nests are tacitly used. Students deserve to share in the mathematical motivation for those choices, and the following four cases are especially revealing.

Case #1: When the x-nest converges to a point which is **fully outside the original function's domain**, the f-induced y-nest, and the f-windows ultimately disappear. Such disappearance can happen also when the x-nest converges to an "infinity". Cases of both kinds are realistic, commonplace, and illuminating — because they disclose that such a convergent x-nest can say nothing about the function's values.

Case#2: When the x-nest converges to a **non-domf point — or to an infinity — that is near to domf** — with each of that nest's intervals comprising points of dom**f, t** he f-values of those dom**f** points define the induced y-nest — and the corresponding nest of f-windows. (This condition allows even for the x-nest to lie fully on just on one side of the x-nest limit.)

Convergence of that x-nest to its limit forces, within the y-nest, a corresponding (non-increasing) progression of f-induced y-intervals — perhaps progressively changing their ends. If any intervals of the y-nest have upper-endpoints, their respective f-windows have top-lines. Convergence of the x-nest causes those upper-endpoints and window-tops to converge, downward, to a number — or to lower infinity, so that the f-window progressively disappears. Likewise, if any of the nested y-intervals have lower-endpoints, their f-windows have bottom-lines. Convergence of the x-nest causes those endpoints and window-bottoms to converge upward, toward a number — or to upper infinity, so that again the f-window progressively disappears.

If any of the nested y-intervals has two endpoints, their respective windows have tops and bottoms.

Convergence of the x-nest causes those endpoints to squeeze toward each other and — since no lower endpoint is above an upper endpoint — ultimately to determine a limiting closed y-interval which might or might not be a member of the y-nest. The associated f-windows converge to a vertical line-segment.

Thus, a non-dom**f** point, p, that is near to dom**f** — although it has no f-value of its own — can gain its own ***limiting interval for f-values***. The same is true of the two "infinities". Such an f-limiting interval (at the limit of the x-nest) expresses the "***clustering tendency***" of the f-values, as the domain and the function are progressively restricted toward that limit.

The f-induced y-nests do not always yield limiting intervals — they might converge to an infinity or to both. But when the convergence is to a y-interval, an upper endpoint of that interval is an **upper limit (or sup) for the f-values from points near p**, ***relative to that x-nest*** — for brevity, "the upper limit of f, at p, or Ulim**f**(p), for THAT x-nest. Likewise for the (inf) lower limit at p, Llimf(p) — and for Ulimf(+∞) and Llimf(+∞), etc. Of course, such upper and lower limits might be equal — for THAT x-nest.

Case#3: When the x-nest converges to a dom**f** point, p, which is **an isolated point of the domain**, the f-window disappears unless p lies within all of the nest's constituent intervals. In the latter case, the point's f-value, f(p) lies within all of the induced y-intervals, and also within the concurrence of the induced y-nest. But the isolation and convergence dictate that there are no other dom**f** points in concurrence of the x-nest. So, the induced y-intervals ultimately become constantly the single-point interval, [f(p)] — and the **f**-windows nest concurs in a single function-point. Of course, that single point says nothing new about the function. In particular, in contrast to case #2, that result says nothing about function-values of the other domf points.

Case#4: When the x-nest converges to a point, p, which is **within domf** — with each of the x-intervals also comprising some non-p points of dom**f** — each of the induced y-intervals comprises f(p) — which then must be within the (closed) limiting interval for f-values. Although that interval loosely expresses the cluster-tendency of f-values from dom**f** points near p, the value, f(p), might seriously distort that picture. For, f(p) might be the sup/inf and far above, or far below, the limiting interval that would result if p were "deleted" from the domain of f. So, the cluster-tendency could be badly obscured by the presence of f(p) within the limiting interval.

Of those four cases, only in case #2 does the function's limiting intervals — and any sup/inf of function-values — at numbers or at infinities clearly express the cluster-tendency of f-values as the domain is restricted. But in that case, while that point or that infinity was near to dom**f**, it was not within dom**f**.

So surfaces the advantage of using delta-nests of delta-intervals. Those enable achievement of limiting intervals even at most points within dom**f** — if those are not isolated points of the domain.

For each real number, p, its deletion from the number line leaves the Dedekind hole, (-∞<u>≤≤</u>p)&(p<u>≤≤</u>+∞). A delta-nest at p uses the p-deleted real-numbers line, and converges to that hole, and consists of intervals which straddle the hole, or which end at the hole. If, at p, the f-induced y-nest converges to an f-values-limit interval, that interval is not affected by whether or not p is within dom**f**. If such an interval occurs at p, an upper end would also be the sup, at p, of the function as restricted to the p-deleted line of real numbers. But without the deletion, when p is within dom**f**, f(p) might be the sup/inf of f, at p — while also differing from the function's upper/lower limits at p (which might not even exist). Of course, the use of delta-nests to define a function's upper/lower limit at a point precludes such limits from occurring at isolated points of the domain (case #3).

The lim operators on functions:

Especially since the advent of the graphics calculator, it far more effective to approach the limits of functions through "zooming windows", than through classical epsilonics. But the option of selecting which x-nests to use concurrently accommodates "left-side" and "right-side" limits at numbers, and "ultimate" limits over the two "infinity" kinds of x-nests. So, the calculator surfaces the fact that nested "spheres" provide the natural context for describing most kinds of limits of transformations of real-coordinate spaces. All the more reason for re-forming the calculus syllabus into the direction of interval-nests.

Nonetheless, even convergent nests and function-windows yield only point-wise definitions for each function's various kinds of limits. But the calculus students' conceptual understanding of a function's limits cannot be adequate until they are fully aware of how a function is changed by passing to each of its various limit-functions — right-upper limit, right-lower limit, etc.

Only by perceiving each of those limiting processes as an ***operator*** that carries a function into a limiting function, can they fully grasp the meaning of ***continuity*** of various parts of a function. The student who cannot draw, upon a function, each of its various limit-functions probably has not yet internalized the concepts.

Unfortunately, the classical symbolism inhibits that perception. Rhetorically, the "lim" passage from "f" to "limf" traditionally is complicated by traditional curricular reliance on "f(x)"which sometimes means the function, f, and sometimes means the composition of f onto a function, x, and sometimes means a point in its range (sometimes even ALL of its range). Curricular reliance on "f(x)" can obscure the picture by requiring use of the expression, "limf(x) at x=3" — classically expressed as "limf(x) as x→ 3". In part because of the confusing use of "f(x)", the latter, cluttered expression fails to exhibit that limf — or even limf(x) is a function in its own right.

In practice, that rhetorical mess presently cannot be avoided — it abounds in the literature. But when it comes to effective instruction, there is no excuse for letting the classical rhetoric of our exploring forefathers to serve as a shackle that impedes learning. Far more effective to use the best possible language for purposes of imparting conceptual understanding, and later to use that language as a tool for digesting the book. Nonetheless, just as when bridging from the rational numbers to epsilon bands, the learning-conscious instructor can provide preparatory supplements.

Note the backwards (English) reading of "log**f**$_{(3)}$" — the choice of 3 leads to whatever number is $f_{(3)}$, whose log-value also is the value of the composite. That convention leads to interpreting "lim**f**$_{(3)}$" as meaning "the limit of the constant, $f_{(3)}$" — which says nothing about limf. But through a nice, clean, and clear variation, an underline may be used, so that "<u>lim**f**</u>$_{(3)}$" can mean the function-value of lim**f**, at 3. Thereby, <u>lim**f**</u> can empower students to later digest, better, the classical expressions.

In the context of function-limits, the bridging over the f(x) confusion begins with re-viewing the "x→p" notion of "x approaches p". From the students' viewpoint, if x is to "approach" anything, x must vary over time — i.e. along a line of real numbers — ultimately converging to a.

To grasp the mathematical substance of that notion requires having a meaning for "a variable converging to a number" (or to an "infinity"). [The classical curricular notion — of x as a letter whose numeric meaning can vary — fails to provide mathematical objects which can be objectively studied.] In decades past, the label, "real variables", was used for what now are called "real functions". The prevalent mathematical model of "variation" is the concept of functions — whose values might vary, over an increasing progression of real-coordinate points. From that viewoint, the notion of a function converging to a number is synonymous with its being asymptotic to a constant.

So arises the instructional advantage of attending, first, the outer limits of functions (i.e. "at infinity"). The notion of "x approaches p" tacitly implies that x is a function which is asymptotic to the constantly-p function — ultimately converging to p, necessarily over an upper-unbounded domain, without ultimately equaling p. A familiar illustration is the language of infinite decimal-points — digit-sequences whose (non-decreasing) decimal series converge to real-valued constant functions. Other examples abound among the combination of algebraic and trigonometric functions. Approaching such asymptotic convergence through a function's upper/lower limits at upper and lower infinities nicely sets the stage for later attending function' sup's and inf's over delta-nests.

For such an asymptotic x, f(x) is a composite function — with f acting as possible, on the range of x. Of course, the case of central interest is where f(x) likewise is asymptotic to some constant, so that limf(x) = L, as x converges to p. But for f(x) to have an upper-unbounded domain, p must be a limit-point of domf. So, it suffices for ran**x** to lie fully within dom**f** — a condition often expressed as "x approaches p over the domain of f".

That "asymptotic" bridge, providing a comprehensible entry into the classical symbolism, gives graphic and precise mathematical meaning to the expression "as x→p, f(x)→L". But such dual convergence does not quite give a viable definition for L being the limit of f, at p. For, depending on the nature of f, and on which p-asymptotic x-variables are used, the f(x) variables might not all converge to L.

Thus it surfaces that underlying the classical expression is a hidden requirement. At p, the limit of f is L iff for **ALL** x which approach p over dom**f**, the composite function, f(x), converges to L. Of course, that condition can be grasped only by examining contra-examples, notably at essential discontinuities.

Such explorations of x→p over dom**f**, with regard for what happens with the f(x) composite, serve more than to enhance digestion of the classical expression of function-limits. They also set the stage for departing from that tradition, to express the limit-operators and their results.

The author and The Institute welcome comments and suggestions which might contribute toward this paper's progressive improvement through subsequent revisions.