MATHEMATICS AS COMMON SENSE: IMAG-INARY COMPLEX NUMBERS

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Abstract:
How does the instructor make mathematical sense of the square roots of negative numbers – for students who already know that all squares are non-negative? Mere postulation of the seemingly impossible contradicts mathematical logic. More "constructively", every 4-quadrant numbers-plane becomes a system of complex numbers, when vector additions and "cyclonic multiplications" are imposed. Thereon, "square root" is simply a collapsing rotation of the plane.

Source:
A major tool of the science of mathematics instructology is the ***syllabus methods*** adaptation of ***critical path methods***, from the managerial sciences. When used in the context of clinical research, syllabus methods uncover many maladies of curricular mathematics syllabi, and ways of improving them. This description of a re-formed syllabus is written to mathematicians who already know the mathematical subject matter, but who are concerned about improving the effectiveness of instruction. Alternative versions for teachers and students are slated for e-publication on http://www.mathsense.org/secrets.htm.

Numbers:
English grammar dictates that "imaginary numbers" constitute a special kind of numbers – which (like "irrational" or "natural" numbers) might or might not be its own **system of numbers**. So, the phrase can make no mathematical sense without a definitive concept of "numbers". For that, we may simply list, as defining postulates, some in-common properties that are enjoyed by each of the various familiar kinds of number systems.

Each system of numbers consists of:
any collection, C, of objects,
onto which are overlaid two functions of two variables,
[call them F(irst) and S(econd) functions], for which:

- both have CxC as their respective domains ("operational completeness"), and
- both have ranges contained in C ("operational closure").
  [So, F and S are called **binary operations** on C.]
- both are commutative
- both are associative
- at least one of them has an operational identity, and
- one which does have an operational identity distributes over the other.
  [It is commonplace to call the distributing operation,
  "multiplication"; and to call the other operation, "addition";
  and to call the multiplication-identity, that

number-system's "1".]

Configurations of number systems

Number systems come in three distinct shapes. The ***linear*** systems are most fundamental, in the sense that the ***circular*** systems (or "modules"), and also the ***planer*** systems (of "complexes"), can be constructed from the linear systems.

Every linear system is *linearly ordered* by any binary relation that is: *reflexive*, and *symmetric*, and transitive. A linear ordering of a system of numbers is a **higher-ordering** (by >) if, for every c in C, (c+1)>c [For brevity, a higher-ordered system of numbers is a **number line**.]

On a number line, there can be no maximum point. There are number lines which have minimum points; and others which do not. Some number lines are ***discrete*** – wherein, every point has a successor. Among those, the Peano lines have minimum points; and the integral lines do not. Some number lines are ***dense*** – wherein every point is a two-directional limit point. Some of the dense number lines (the "re(g)al" or "re-al" ones) also are solid – but some dense lines are "infinitesimal sieves", having dense lines of (Dedekind) ***holes***.

Their planes:

Each number line, (**L**,>,+, .), generates its own plane – as the (LxL) set of binary points (a,b) from L. Each *numbers-plane* becomes *a system of complex numbers*, when **vector additions** and **cyclonic multiplications** are imposed upon that plane – providing that those operations satisfy the postulates which define number systems.

So, in some cases, numbers-planes can become systems of **complex numbers** – and in some cases, they cannot. Each line of ***whole*** numbers generates a discrete ("pegboard") plane that has only one quadrant. Its points admit to vector addition, and to scaler-multiplication (by whole numbers). But that plane also is ***operationally-incomplete*** for vector subtraction – if a pair of distinct non-origin points can be vector-subtracted, that pair's commute cannot be subtracted. That flaw is enough to preclude the whole-numbers plane from becoming a system of complex numbers.

However, each line of ***integers*** likewise generates a discrete plane – but one that has *four* quadrants. It, too, admits to vector addition, and to scaler multiplication (by integers), but it also is operationally complete for vector subtraction. That condition allows for planer points to be "multiplied" with each other, "in the complex mode".

The last example logically destroys the prevailing curricular myth that "*the*" system of complex numbers includes "the" system of real numbers. Every four-quadrant plane that is generated by a linear system of numbers becomes its own system of complex numbers, as soon as vector addition and cyclonic (or "complex") multiplication are imposed.

Obviously, the discrete case of ***the integers-plane*** is the most viable context for initial learning and instruction about systems of complex numbers. But with that kind of simplicity, an introduction to complex numbers can be simply a "weather channel" topic, even for the middle grades.

The integers-plane also serves to clarify still another traditional curricular confusion. In the curricular theories of numbers, "positive numbers" are defined as being the above-zero points on whatever number-lines also have below-zero points – and the latter points are that line's "negative numbers". However, the four-quadrant planes do not have "above zero" or "below zero" points. Instead, they have four axial-directional rays from the origin – *forward, backward, upward, and downward* . So, each planer point lies in one or two directions from the origin.

Thus, if "positive" means "above zero" – with the plane's vector-zero being the (0,0) origin – the "above zero" points , if anything, would have to constitute the upward ray from the origin. In one context, the points along the origin's upward ray do behave as positive numbers should .But in the context of complex numbers, they do not. So, in that above/below meaning, there can be no (above-zero) no "positive" complexes – and so no "negative" complexes – and so no "square roots of negatives".

[Of course, a differing definition of "positive" can yield a very different conclusion. At higher levels of mathematical maturity, a number-system's "positive numbers" may postulationally be defined as ***the (+,•) operational-closure of that system's 1***. In that sense, even a system of natural numbers has a system of "positives", even though it has no zero or negatives. In the same sense, the complex plane's forward ray from the origin would qualify as being that number system's "positive numbers" – and their negatives would qualify as being the planer system's "negative numbers". But it is hardly realistic to expect students within the core-curriculum, to have the sophistication needed for so re-defining their longstanding and familiar terminology.]

Imaginary and objective numbers:
Coordinate planes are routinely used for graphing functions whose domains and ranges lie within the same number line. But sloppy exposition quite obscures how the graphing actually works.

If a number-line, L, has a zero, the subset consisting of all (x,0) points is that plane's *first axis* (variable in the first place; constantly 0 in the second place); and the subset consisting of all (0,x) points is that plane's *second axis*

[Note that the x can occur in either place. More generally, for each planer point, (x,y), its commute, (y,x), also is a planer point – as are (x,x) and (y,y). So, it is mathematically absurd to insist that "the x is on the x-axis" – or to use any letter as the label for an axis, while concurrently using it also as a numeric variable. Of course, that curricular confusion commonly confounds students.]

Each of those two axes is order-isomorphic to whatever number-line, L, generates that plane. Also, each axis is closed for vector addition – and also for place-wise multiplication. So, with respect to those two "natural" vector operations, each axis also is ***operationally*-isomorphic** to L. So, imposing those "natural" operations along each axis makes each into a number system of the same kind as L. [That is the sense in which, as earlier mentioned, the up-from-the-origin points might be regarded as (above-zero) "positives".]

In the sense of isomorphism, all three number lines – the plane-generating line and its two coordinate axes – are "*abstractly identical* ". That condition allows us the very convenient, logical sloppiness of regarding the domain of a function as lying along the first axis, and its range as lying along the second axis.

For example, for x^2: 3→9, it is commonplace to regard 3 as an ***object*** from the domain, and 9 as its ***image*** within the range – and to use the (3,9) point to present that 3→9, mapping. However, if we get quite picky about how planer graphing of functions actually works, we find that the (3,9) point is actually presenting the mapping, (3,0)→(0,9) – from the first axis, into the second axis. [The traditional curricular failure to surface that difference routinely muddies the students' initial learning of the graphing process.]

So, as the graphing goes, the ***object*** is (3,0); and its ***image*** is (0,9). From that viewpoint, the first axis is the ***object axis***; and the second axis is the ***image axis***. So, due to the isomorphisms, the points of the first axis are the ***object-numbers*** (**or "objective" numbers**) from the underlying number-line – and the points of the second axis are its ***image-numbers*** (***or "imaginary" numbers***). As soon as the student is introduced to the planer *graphing of functions*, the two axes may rightfully be called "***the objective axis***" and "***the imaginary axis***".

Teachers who have been reared through the mystical notions of square-roots of negative numbers (clarified, below) – and who have never grown beyond that level – might regard such equating of "imaginary numbers" with the "y-axis", as being intolerable heresy. Pity their students. But teachers who have gone through graduate courses in complex functions might, instead, regard the axial interpretation, as being merely an innocuous twist of a more familiar interpretation. But those who are aware of how often students are confounded by the irrational curricular postulation of square roots of negative numbers can soon appreciate the gain in conceptual understanding.

This axial notion of ***objective numbers and imaginary numbers*** is tacitly, but sub-versively woven through the graphing of functions. Quite notably, it has ***no intrinsic connection*** with complex numbers except that all three number systems (objective, imaginary, and complex) arise from the number line's coordinate plane. Of course, a lower-unbounded number line's imaginary numbers (along the second axis of its plane) also become its ***imaginary complexes*** (on that plane) – *after* the complex operations are imposed.

That axial number-lines interpretation serves two important ***instructive*** purposes. First, it aids students to perceive the schema for graphing functions – by complementing the "run-elevate" mode of plotting function-points. Second, its independence from the context of complex numbers gives an appropriate, but rationally plausible meaning to "imaginary" numbers – which later equates with the meaning of "imaginary complexes" – while precluding subsequent misconceptions about imaginary numbers being "fictitious numbers" or non-existent mathematical non-senses.

True, systems of complex numbers go a step beyond graphing, by regarding the coordinate plane as a ***2-dimensional vector space***. In that context, each of the two axes is a vector subspace consisting of all scaler multiples of its unit vector. The (0,1) unit of the image-axis is routinely called ***i*** – whereby, the second axis is better called the ***i-axis***, than the "y-axis", since the "y" also is used as a variable. In the other axial direction, the (1,0) unit of the objective axis best not be called "***o*** " – else it could be confused with zero. But again with regard for graphing, the (a,b) function-point can be plotted by ***r***-unning from the origin, to (a,0) – where adding-on (0,b) sets the ***i***-nclination of the radius from the origin, to (a,b). In that context, each (a,b) planer point can be ***vector-analyzed*** as a***r***+b***i*** – where the "***r***" also can later be called the (run) unit of the "***r***eal axis". [To avoid a conflict, the length of a vector's ***r***adius – often called "r" – can be called its ***modulus, m***.]

Once complex-multiplication has been imposed onto the plane, the ***r***-unit also serves as the "1" of the complex numbers. Unfortunately for students, that (1,0) complex "1" is easily confused with the "1" of the plane-generating number line. In a sense, that confusion conveniently justifies using the a+b***i*** formulas for points in the coordinate planes. But those formulas give the novice the impression that the (b***i***) vectors are being added to the (a) scalers – rather than regarding the a+b***i*** formulas as speaking of points on the plane. Far better for the student, if the instructor sustains the vector-format, a***r***+b***i***, until its planer meaning is so well understood that even the un-written complex-1 is known to be the (1,0) pair of scalers.

The unary operations:
The traditional curricular fetish with *binary* addition and multiplication inherits from the rhetorical *elegance* of professionally defining number systems in terms of binary operations – as is done at the beginning of this paper. But as in a myriad of other settings, such elegance can be at the expense of failing to impart conceptual understanding of the complex operations. For, reliance on binary operations badly muddies human comprehension of the nature of the numeric operations. [In psychomathematical perspective, the student who would regard the operations as being of the binary kind must sense the 3-dimensional nature of the x+y function, and of the xy function.]

The well known alternative of defining groups and rings in terms of ***unary*** operations is a bit more messy. . However, the unary operations also can be simply derived from the binary cases. The binary-addition operation has its own lateral cross-sections – in which the second place is held constant, as a ***parameter***. Thereby is defined, for each number, a, the T_a ***translation***, x+a – which is a *unary* addition on C.

For ***linear*** systems of numbers, those translations can be graphed in their respective coordinate planes. But of even greater conceptual importance is that each translation of a number-line can be ***abstractly*** envisioned ("slide-rule" wise) as concurrent ***movements of all points*** along the line of numbers. Likewise, cross-sections of the binary-multiplication operation define the P_a ***proportions***, a•x – each of which stretch or shrink the entire line, depending on the magnitude of a.

The same "unary functions" principle of instruction is of even greater conceptual importance with ***planer*** systems of numbers. For, on the plane, each binary operation is 6-dimensional, linking an ordered pair of 2-place points to a third 2-place point. Thereon, binary-addition of vectors is commonly presented through parallelograms. That suffices, when adding only a specified pair of vectors. But it is difficult to abstractly envision all such parallelograms as concurrently working on all pairs of planer points. However, for each *unary* vector-addition, the parameter-vector is fixed – and the translation concurrently moves all points of the plane, simply along parallel, same-length arrows.

Also on that vector 2-space, each ***scaler***-multiplication (by a number from the underlying linear system) is necessarily a unary function on the plane. It simply proportions the planer points' radii (and their ***m***oduli) – moving all points toward the origin, or away from it, depending on the scaler. But those scaler-multiplication functions also constitute a prelude to unary complex-multiplication of planer points.

The 6-dimensional binary-***multiplication*** of complexes is even more difficult to envision, than is the binary addition. Admittedly, when two a***r***+b***i*** complexes are re-coded in exponential form – $me^{\Theta i}$ – it is more readily seen that complex-multiplication both scaler-multiplies the two moduli, while adding their two angles. But even in that context, the unary operations – each using a fixed $me^{\Theta i}$ to multiply all other complexes – once again reveal what actually is going on. The fixed radius multiplies the radii of all planer points – exactly as does each scaler-multiplication of the plane – while the fixed angle rotates the entire plane. In the combination, all planer points are turned around the origin – and in the process, all are thrown outward, or all are thrown inward, or all preserve their distance from the origin, depending on the modulus of the multiplier.

Thus, each unary complex-multiplication is ***cyclonic*** throughout the coordinate plane – a condition that is not readily perceived through focusing on binary-multiplication of complexes. So, the squaring of complexes doubles the angle of every non-origin point – the greater a point's angle, the more it is rotated. Meanwhile, it squares each point's modulus – so preserving the modulus for points on the unit circle, while sending each of the other points further away from the unit circle.

Inversely, the (primary) square-root function halves the angles of all planer points, so carrying the unit circle onto its upper semi-circle. In particular, ***i*** is the complex-multiplication's ***cyclonic square root of the planer point, (-1,0)*** . In the same (advanced) sense that (1,0) is the "1" of "the positive set" of complexes, the (-1,0) is a "-1" of the (positive set's) "negative" complexes. Accordingly, with respect to complex multiplication of planer points, ***i*** is the cyclonic square root of the ***objective*** "-1" – which authors and teachers routinely confuse with the generating number-line's own multiplication's (impossible) square-root of the scaler, -1. The student who is not confounded by such mathematical non-sense simply is not thinking!

A constructive syllabus:
The mathematical non-sensibility of the traditional curricular syllabus through the topic of complex numbers stems from postulating the existence of the logically impossible number-line square root of the number-line's negative 1 – whose scaler multiples are construed as "imagin-ary" ***fictitious*** numbers – which are mysteriously added to genuinely "real" numbers – to produce conceptually complexified ***"ethe-real numbers"***. Nonetheless, it is commonplace to interpret complex-addition as a binary operation among planer vectors. If the same were done for complex-multiplication, the syllabus could nicely circumvent postulation of the impossible. The resulting re-formed syllabi through the complex numbers can be mathematically common-sensible to anyone who has grasped the bare essentials of basic algebra (1).

The crux of the common-sense re-formation is the ***reconstruction of complex multiplication*** - as unary operations that apply throughout the 2-dimensional vector space. The curricular tradition has been simply to arbitrarily inject the formal definition of complex multiplication – devoid of conceptual motivation – so making that operation appear to be quite mysterious. What would ever prompt anyone to concoct any such a binary operation? What on earth does it mean? Fortunately, the ***vector theory*** provides a more reasonable, practical, and common-sensible way to achieve complex multiplications, powers, and roots.

The vector journey into complex multiplication begins with the discrete, 2-coordinate plane generated by the integers – and with its use in the graphing of functions. Within that context, the graphing uses one axis as objective-integers, and one as imaginary-integers – if only to later circumvent the prevalent myth about any (fictitious) "square root of (the integer) negative-one".

That integers plane is re-visited as a 2-dimensional vector space. Thereon, its binary vector-addition manifests as the parallelogram law, and generates its unary-additions as translations of the plane. The (unary) scaler-multiplications of vectors (by integers) are carefully examined with regard for the various ways in which they proportion the plane – depending on their respective parameters.

For conceptually deriving complex multiplication (from which follows the powers and roots), the critical factor is that, for each ar+bi vector, the (a,b) coefficients express the (run**:**elevate) slope of that vector's ray from the origin – even for the imaginary numbers, which have no ***slope-number***.

That fact allows us to identify, for each non-origin, ar+bi vector, *its* 90^o *rotation*,-b***r***+a***i*** – herein called the ***cohort*** of ar+bi. Of course, each planer vector has the same (Pythagorean) length as its cohort. Moreover, each such pair of vectors spans the plane – i.e. for each planer vector, **v**, there exists a scalers-pair, (c,d), for which

v = c(ar+bi)+d(-b ***r***+a ***i***).

Of course, the covering theorem is that ***every*** non-colinear pair of non-origin vectors so spans the plane. One formal proof is simply a matter of solving a pair of simultaneous equations. But its conceptual understanding comes through the graphics of randomly choosing a ("basis") pair of vectors – and randomly choosing other vectors – to graphically achieve their respective parallelogram analyses in terms of the chosen basis.

When an (a***r***+b***i***) vector and its (-b***r***+a***i***) cohort are used as a ("square") basis, its c(ar+b***i***)+d(-b***r***+a***i***) analysis of each planer vector presents a *rectangle* – whose vector-sides are scaler multiples of the two, perpendicular basis-vectors. The resulting (c,d) pair of scalers thus combines with each (a,b) point – which also is a pair of scalers – according to a rule**: multiply (a,b) by c; and multiply its (-b,a) cohort by d; and (diagonally) add those two products**. Since that mode of combination entails scaler multiplications of a vector and of its cohort – albeit also invoking addition – it is realistic to regard that mode as a way of "multiplying" points on the plane.

Thereby, (c,d)◦(a,b) [read "c-by-d of ab"] means c(a***r***+b***i***)+d(-b***r***+a***i***). Since all scaler multiplications distribute, c(a***r***+b***i***)+d(-b***r***+a***i***) = (ca***r***+cb***i***)+(-db***r***+da***i***) = (ca-db)***r***+(cb+da)***i***. Of course, that last formula – (c,d)◦(a,b) = (ca-db)***r***+(cb+da)***i*** – is the traditional definition for "complex multiplication" of points of the plane – which is viable only for 4-quadrant planes. But until the students have well internalized ***the rectangular nature of complex multiplication*** , instruction more effectively regards "(c,d) ***of*** (a,b)" as meaning c times (a,b), plus d times its cohort.

Thus, just as the complex addition and subtraction of planer points have parallelogram interpretations, complex multiplication of planer points has a rectangular interpretation – and it rarely, if ever, is provided in traditional texts. But the same is true of ***complex division***, as follows.

Note that our conceptual derivation of complex multiplication came from the space-spanning theorem. Within the resulting context of complex multiplication of planer points, the rectangular spanning theorem now says that, for each non-origin point, (a,b), there is, for every vector, **v**, a (c,d) point for which **v** = (c,d)◦(a,b). By so applying (a,b) to **v**, to get (c,d), we have "divided" **v** by (a,b), to get (c,d) as a "quotient". For example, for (a,b) = (3,5), and v = (-23,41), the (c,d) quotient can be got from [3c-5d = -23] and [5c+3d = 41]. Thereby, it is seen that v = 4(ar+bi)+7(-b ***r***+a ***i***) – and so, **v** ÷ (a,b) = (c,d) = (4,7). It also means that every vector can be "factored", using any chosen vector as one of the factors.

Thus, each non-origin vector (with its cohort) divides every planer vector – and the quotient is the pair of scaler multipliers for the divisor and its cohort. The traditional "conjugate" process for dividing complexes is thus seen as an alternative means of achieving a rectangular analysis of the dividend-vector, in terms of the divisor-vector and its cohort.

The cyclonic nature of complex multiplications is best seen through a *similar rectangles theorem* which pertains to each unary complex-multiplication. In those, the "multiplier" is fixed as a parameter – e.g. as (c,d) = (3,4). In that case, every ar+bi vector is tripled, and its cohort vector is quadrupled, to give its 3(a***r***+b ***i***)-by-4(-b***r***+a ***i***) rectangle. The (c,d)◦ operation thus converts each ar+bi vector into its

3(ar+bi)+4(-b ***r***+a ***i***) diagonal. So, the (3,4)◦ multiplication increases the modulus of each planer vector, by a factor of 5.

In general, all such (c,d)◦ rectangles are geometrically similar. That similarity enables Pythagorean calculations to confirm that each (c,d)◦ operation multiplies all moduli, throughout the plane, by the same number [even though that number might not be among the scalers for that particular 2-space] – and that the modulus of (c,d)◦(a,b) always is the product of the modulus of (a,b), with the modulus of (c,d). That theorem provides still further justification for pinning the "multiplication" label onto the (c,d)◦ operations.

Also because of that similarity condition, the (c,d)◦ rotation-angle, from (a,b) to (c,d)◦(a,b), is the same for all (a,b) planer points. In particular, the rotation-angle from ***r*** to (c,d)◦***r*** is defined to be ***the angle*** *of (c,d)*. Thus, the (c,d)◦ multiplication rotates all vectors by the angle of (c,d). It follows that the angle of (c,d)◦(a,b) is the sum of the (a,b) angle and the (c,d) angle.

In the combination, complex-multiplying by (c,d) turns the entire plane around the origin, through the angle of (c,d) – while multiplying all radii by the modulus of (c,d). In particular, every unary complex-multiplication by a *point on the unit circle* simply rotates the entire plane by its own angle. In the extreme case of the integers-plane, the unit circle consists of only four points: ***r*** and-***r***, and ***i*** and -***i***.

One special case – which is essential, for it to qualify as a "multiplication" – is the source of longstanding curricular confusion about complex numbers, their products, their powers, and their radicals. Since ***r*** = (1,0), it follows that ***r*** ◦(a,b) = (a,b) – which also is (a,b)◦***r***. So, ***r*** is the multiplicative "1" for complex multiplication. Accordingly, its negative, -***r***, is (-1,0) – which serves as a complex "-1". But the traditional mistaking of those two planer points as being their scaler counterparts is best avoided by persistently adhering to the use of ***r*** and-***r***.

For complex multiplication, the multiply-moduli&add-angles theorem sheds much light on the *rational powers* of planer points. In particular, when the plane's generating number-line is sufficiently rich, every planer vector has at least one complex square-root.

Successive whole-number powers of a chosen non-objective vector generate a counter-clockwise ***power coil*** around the origin. If that vector's modulus is higher than 1, the coil diverges, outward. If its modulus is lower than 1, the coil converges to the origin. If its modulus is 1, the coil lies on the *unit circle* .

Inversely, successive negative-integer powers [repeated divisions of ***r***, by the exponential base] generate clockwise coil-tails which diverge, converge, or not, depending on the divisor's modulus. All existing radical-roots of a chosen complex lie along such a coil. However, on the plane of integer-complexes or of rational-complexes, many vectors lack many nth roots. Of course, all existing nth roots have all whole-number powers, so generating their own coils. In that sense, a chosen vector might have many non-whole rational powers – or only whole-number powers.

Since ***i*** = 0,1), it follows that ***i***◦(a,b) = (-b,a) = (a,b)◦***i*** – which says that complex multiplication-by-***i*** rotates all planer points 90°, into their cohorts. In particular, the whole-number ***i***-powers – (***i***◦)^2, (***i***◦)^3, (***i***◦)^4, (***i***◦)^5, etc. – reveal that each of the integers-plane's unit-circle points have some nth roots, but not others.

Notable among those roots is that – since ***i*** ^{2} = -***r*** – one complex-sqrt(-***r***) = ***i***. [*The other* "square root of (complex)-1" is- ***i***.] The traditional curriculum routinely misconstrues that " ***i*** = sqrt(-***1***)" theorem as speaking of the *number-line's* square root of the number-line's (-1) – which is logically impossible, whatever the underlying number-line.

Of geometric interest is that, when the underlying number-line is dense, so is its plane's unit circle. It means that its unit circle can serve as a radius-1, circular protractor for assigning numerical measures to the angles of all vectors. Of course, such a device is fully accurate only for those angles whose moduli can be reduced to 1, through division by whatever scalers generate the plane. Otherwise, one must settle for approximations of angle measures.

Scalers permitting, the angle of the primary nth root of ***i*** is the counter-clockwise one-(4n)th of the unit circle – and its first 4n whole-powers fraction the unit circle into 4n arcs. When that works for all n, the non-negative rational powers of ***i*** constitute a dense subset of the unit circle. Such a protractor suffices for at least an approximate measure of each vector's angle, in terms of $i^{p/q}$. But since that *quadrant-protractor* speaks only of approximations, one might as well describe the angles as decimal powers of ***i*** – as with $i^{2.7936482}$ depicting the third quadrant angle of a large power of the (10^7)th root of ***i***.

The author and The Institute welcome comments and suggestions which might contribute toward this paper's progressive improvement through subsequent revisions.