Deterministic and Stochastic Models for the Spread of Fear and Opinions

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History and Motivation

Who, When, Where, Why

- Reanne Bowlby, Ange Sinamenye, Elizabeth Steen, Cameron Swofford, and Katie Vinopal, with LT Grant

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- Topic is models of social interaction in populations, but the methods are generally useful in mathematical biology, including modeling, Markov processes and nonlinear differential equations.

Imagine a town in which some “regular” citizens live in fear of the local **M**ob, while others have confidence in the **P**olice force, and remain unafraid.

Individuals from each group encounter other individuals, independent of the group.

The model assumes a throughly mixed population with uniform probability of an encounter between two individuals providing the random interaction, regardless of group.

In conversation, a fearful person may convince someone unafraid that the Mob is dangerous, or the unafraid may convince the fearful individual that the Police can protect them.

What will the general opinion of the town be as time goes on?

- **Two-Party Politics.** One political party (M) spreads one view; the opposed party (P) spreads another. Each person in the general populace leans toward one party or the other.
- **Advertising and Public Opinion.** Two soft drinks vie for brand loyalty in a population. The roles of those who convince the population to each side would be taken to be sales agents for each brand.
- **General Debate.** This model can represent any two opposing viewpoints, not necessarily related to fear. We refer to members of M and P as zealots because they are of fixed opinion.

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Our variables of interest, in the stochastic analysis of our model, are:

- f_k, u_k - The number of fearful and unafraid members of the population at time step k.
- M, P - The number of M and P agents. These numbers are fixed, and assumed to be positive.
- α - The chance that a $\langle u, f \rangle$ encounter produces 2 f's. And similarly, $(1 - \alpha)$ is the chance we end this time step with 2 u's.

- $\langle u, M\rangle$ denotes the interaction between an unafraid person and a Mob agent so that $(f_{k+1}, u_{k+1}) = (f_k + 1, u_k - 1)$
- $\langle f, P\rangle$ denotes the interaction between a fearful person and a Police agent so that $(f_{k+1}, u_{k+1}) = (f_K + 1, u_k - 1)$.
- $\langle u, f\rangle$ denotes the interaction between an unafraid person and a fearful person so that

$$\langle(f_{k+1}, u_{k+1}) = \begin{cases} (f_k + 1, u_k - 1) \text{with probability } \alpha \\ (f_k - 1, u_k +) \text{with probability } 1 - \alpha \end{cases}$$

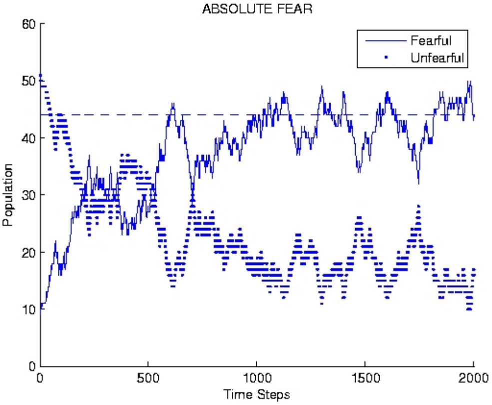


Figure: Simulation: $u_0 = 50$, $f_0 = 10$, $M = 5$, $P = 5$, $\alpha = .6$, One Trial,

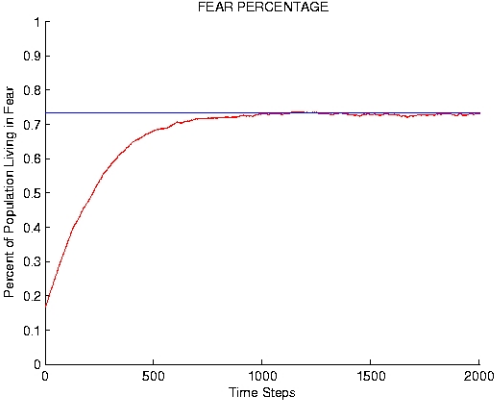


Figure: Simulation: $u_0 = 50$, $f_0 = 10$, $M = 5$, $P = 5$, $\alpha = .6$, Average of 500 Trials

With scaling, substitution, and simplification the equation becomes

$$\dot{f} = uM - fP + \alpha uf - (1-\alpha)uf.$$

where $\dot{f}$ represents the rate of change with respect to τ.

Using the same assumptions and scaling for u, we have

$$\dot{u} = -uM + fP - \alpha uf + (1-\alpha)uf.$$

Note that $\dot{f} + \dot{u} = 0$, since $u + f$ is constant. We substitute $u = 1 - f$ and let $\beta = 2\alpha - 1$ to obtain

$$\dot{f} = -\beta f^2 + (\beta - P - M)f + M.$$

When $\alpha \neq 1/2$, get a Ricatti equation

$$f(t) = \frac{\delta}{\beta} + \frac{\gamma}{\beta}\left(\frac{e^{\gamma t} - Ce^{-\gamma t}}{e^{\gamma t} + Ce^{-\gamma t}}\right) \qquad (1)$$

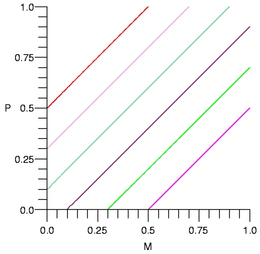
where

$$\begin{aligned} \delta &= \frac{(\beta - P - M)}{2} \\ \gamma &= \frac{1}{2}\sqrt{2M\beta + \beta^2 - 2\beta P + P^2 + 2PM + M^2}, \\ C &= \frac{-\delta - \gamma + \beta f(0)}{\delta - \gamma - \beta f(0)}. \end{aligned}$$

$$\lim_{t\to\infty} f(t) = \frac{\delta}{\beta} + \frac{\gamma}{\beta} = F(M, P, \alpha). \qquad (2)$$

Winning the Argument

We define winning as swaying at least 50 percent of people to "fearful." Each diagonal line represents a different α level. At a particular α level, each line represents (M, P) pairs such that $F(M, P, \alpha) = .50$, that is a 0.50-level curve in (M, P) space for a fixed value of the parameter α. Anywhere to the left of the line gives $F(M, P, \alpha) < .50$ and anywhere to the right gives $F(M, P, \alpha) > .50$.



$$
\begin{aligned}
\mathbb{E}[f_{k+1}|f_k] &= f_k + \Pr[\langle u, M\rangle]\cdot(1) + \Pr[\langle f, P\rangle]\cdot(-1) \\
&\quad +\alpha\Pr[\langle f, u\rangle]\cdot(1) + (1-\alpha)\Pr[\langle u, f\rangle]\cdot(-1) \\
&= f_k + 2\left(\frac{u_k}{T}\right)\left(\frac{M}{T-1}\right) - 2\left(\frac{f_k}{T}\right)\left(\frac{P}{T-1}\right) \\
&\quad +2\alpha\left(\frac{u_k}{T}\right)\left(\frac{f_k}{T-1}\right) - 2(1-\alpha)\left(\frac{u_k}{T}\right)\left(\frac{f_k}{T-1}\right) \\
&= f_k + 2\left(\frac{G-f_k}{T}\right)\left(\frac{M}{T-1}\right) - \left(\frac{f_k}{T}\right)\left(\frac{P}{T-1}\right) \\
&\quad +2(2\alpha-1)\left[\left(\frac{G-f_k}{T}\right)\left(\frac{f_k}{T-1}\right)\right] \\
&= \xi\left[-\beta f_k^2 + \left(\beta G - M - P + \left(\frac{T(T-1)}{2}\right)\right) f_k +
\end{aligned}
$$

Equilibria

two points of equilibria:

$$V_1 = \frac{G\beta - (M+P) + \sqrt{(M+P)^2 + 2G\beta(M-P) + G^2\beta^2}}{2\beta},$$
$$V_2 = \frac{G\beta - (M+P) - \sqrt{(M+P)^2 + 2G\beta(M-P) + G^2\beta^2}}{2\beta}.$$

It can be shown that if $MP > 0$, then $0 \leq V_1 \leq G$ and $V_2 \geq G$ when $\alpha < 1/2$ and $V_2 \leq 0$ when $\alpha \geq 1/2$.

Stability

If $MP > 0$ and $T > 5$, then V_1 is asymptotically stable and V_2 is unstable. Furthermore, the sequence f_k is monotone, tending to V_1.