Deterministic and Stochastic Models for the Spread of Fear and Opinions

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History and Motivation
Who, When, Where, Why

- Reanne Bowlby, Ange Sinamenye, Elizabeth Steen, Cameron Swofford, and Katie Vinopal, with LT Grant
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- Topic is models of social interaction in populations, but the methods are generally useful in mathematical biology, including modeling, Markov processes and nonlinear differential equations.
Imagine a town in which some “regular” citizens live in fear of the local Mob, while others have confidence in the Police force, and remain unafraid.

Individuals from each group encounter other individuals, independent of the group.

The model assumes a thoroughly mixed population with uniform probability of an encounter between two individuals providing the random interaction, regardless of group.

In conversation, a fearful person may convince someone unafraid that the Mob is dangerous, or the unafraid may convince the fearful individual that the Police can protect them.

What will the general opinion of the town be as time goes on?
Model
Other interpretations of the model

- **Two-Party Politics.** One political party (M) spreads one view; the opposed party (P) spreads another. Each person in the general populace leans toward one party or the other.

- **Advertising and Public Opinion.** Two soft drinks vie for brand loyalty in a population. The roles of those who convince the population to each side would be taken to be sales agents for each brand.

- **General Debate.** This model can represent any two opposing viewpoints, not necessarily related to fear. We refer to members of M and P as zealots because they are of fixed opinion.
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- $f_k, u_k$ - The number of fearful and unafraid members of the population at time step $k$.
- $M, P$ - The number of $M$ and $P$ agents. These numbers are fixed, and assumed to be positive.
- $\alpha$ - The chance that a $\langle u, f \rangle$ encounter produces 2 $f$’s. And similarly, $(1 - \alpha)$ is the chance we end this time step with 2 $u$’s.
Model Interactions

- $\langle u, M \rangle$ denotes the interaction between an unafraid person and a Mob agent so that $(f_{k+1}, u_{k+1}) = (f_k + 1, u_k - 1)$
- $\langle f, P \rangle$ denotes the interaction between a fearful person and a Police agent so that $(f_{k+1}, u_{k+1}) = (f_K + 1, u_k - 1)$.
- $\langle u, f \rangle$ denotes the interaction between an unafraid person and a fearful person so that

\[
\langle (f_{k+1}, u_{k+1}) \rangle = \begin{cases} 
(f_k + 1, u_k - 1) & \text{with probability } \alpha \\
(f_k - 1, u_k + 1) & \text{with probability } 1 - \alpha 
\end{cases}
\]
Model

One Simulation

Figure: Simulation: $u_0 = 50$, $f_0 = 10$, $M = 5$, $P = 5$, $\alpha = .6$, One Trial,
The Spread of Fear and Opinions

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Large Population Approximation

Small Population Mean

Figure: Simulation: $u_0 = 50, f_0 = 10, M = 5, P = 5, \alpha = .6$, Average of 500 Trials
Large Population Approximate Deterministic Equation
Derivation and Scaling

With scaling, substitution, and simplification the equation becomes

\[ \dot{f} = uM - fP + \alpha uf - (1 - \alpha)uf. \]

where \( \dot{f} \) represents the rate of change with respect to \( \tau \).

Using the same assumptions and scaling for \( u \), we have

\[ \dot{u} = -uM + fP - \alpha uf + (1 - \alpha)uf. \]

Note that \( \dot{f} + \dot{u} = 0 \), since \( u + f \) is constant. We substitute \( u = 1 - f \) and let \( \beta = 2\alpha - 1 \) to obtain

\[ \dot{f} = -\beta f^2 + (\beta - P - M)f + M. \]
Solving the Deterministic Equation and its Meaning

When $\alpha \neq 1/2$, get a Ricatti equation

$$f(t) = \frac{\delta}{\beta} + \frac{\gamma}{\beta} \left( \frac{e^{\gamma t} - Ce^{-\gamma t}}{e^{\gamma t} + Ce^{-\gamma t}} \right)$$

(1)

where

$$\delta = \frac{(\beta - P - M)}{2},$$

$$\gamma = \frac{1}{2} \sqrt{2M\beta + \beta^2 - 2\beta P + P^2 + 2PM + M^2},$$

$$C = \frac{-\delta - \gamma + \beta f(0)}{\delta - \gamma - \beta f(0)}.$$ 

$$\lim_{t \to \infty} f(t) = \frac{\delta}{\beta} + \frac{\gamma}{\beta} = F(M, P, \alpha).$$

(2)
We define winning as swaying at least 50 percent of people to “fearful.” Each diagonal line represents a different $\alpha$ level. At a particular $\alpha$ level, each line represents $(M, P)$ pairs such that $F(M, P, \alpha) = .50$, that is a 0.50-level curve in $(M, P)$ space for a fixed value of the parameter $\alpha$. Anywhere to the left of the line gives $F(M, P, \alpha) < .50$ and anywhere to the right gives $F(M, P, \alpha) > .50$. 

![Graph showing the spread of fear and opinions]

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**Small Population Mean**
Stochastic Model
Equation for the Mean

\[
\mathbb{E}[f_{k+1}|f_k] = f_k + \Pr[\langle u, M \rangle] \cdot (1) + \Pr[\langle f, P \rangle] \cdot (-1) \\
+ \alpha \Pr[\langle f, u \rangle] \cdot (1) + (1 - \alpha) \Pr[\langle u, f \rangle] \cdot (-1) \\
= f_k + 2 \left( \frac{u_k}{T} \right) \left( \frac{M}{T-1} \right) - 2 \left( \frac{f_k}{T} \right) \left( \frac{P}{T-1} \right) \\
+ 2\alpha \left( \frac{u_k}{T} \right) \left( \frac{f_k}{T-1} \right) - 2(1-\alpha) \left( \frac{u_k}{T} \right) \left( \frac{f_k}{T-1} \right) \\
= f_k + 2 \left( \frac{G-f_k}{T} \right) \left( \frac{M}{T-1} \right) - \left( \frac{f_k}{T} \right) \left( \frac{P}{T-1} \right) \\
+ 2(2\alpha-1) \left[ \left( \frac{G-f_k}{T} \right) \left( \frac{f_k}{T-1} \right) \right] \\
= \xi \left[ -\beta f_k^2 + \left( \beta G - M - P + \left( \frac{T(T-1)}{2} \right) \right) f_k + \right]
\]
two points of equilibria:

\[
V_1 = \frac{G\beta - (M + P) + \sqrt{(M + P)^2 + 2G\beta(M - P) + G^2\beta^2}}{2\beta},
\]

\[
V_2 = \frac{G\beta - (M + P) - \sqrt{(M + P)^2 + 2G\beta(M - P) + G^2\beta^2}}{2\beta}.
\]

It can be shown that if \( MP > 0 \), then \( 0 \leq V_1 \leq G \) and \( V_2 \geq G \) when \( \alpha < 1/2 \) and \( V_2 \leq 0 \) when \( \alpha \geq 1/2 \).
If $MP > 0$ and $T > 5$, then $V_1$ is asymptotically stable and $V_2$ is unstable. Furthermore, the sequence $f_k$ is monotone, tending to $V_1$. 