A STANDARDIZED LOGIC NOTATION FOR EVERYDAY CLASSROOM USE

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For students studying undergraduate mathematics, it is common and proper to encounter logical symbols, such as connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, in courses specifically containing logic and truth tables as topics, in other courses relying on the ability to manipulate these symbols, or just incidentally as an instructor uses them to help communicate mathematical ideas in lectures for any course. In countries outside the United States, it is not unknown for some introduction of these symbols, along with truth tables, to be standard by the end of secondary school. The usefulness and beauty of symbolic logic often motivate U.S. instructors to introduce the subject at some low level to students, even in the secondary schools, though it is rare for the actual classroom texts to discuss abstract, symbolic logic except in courses involving proofs, and then only well into the university level. In particular, algebra and calculus texts almost completely avoid the issue, apparently instead appealing to the students' "common sense."

The main purposes of this paper are two-fold. The first purpose is to describe a simple and intuitive standardization of logic notation, both for foundational courses which develop symbolic logic, and for general, everyday use in undergraduate mathematics courses. Second is an argument that undergraduate mathematics that does not usually include logic formally could benefit from widespread use of symbolic logic, particularly if it is standard and thus truly portable.

Done well, the use of symbolic logic can bring much needed clarity to a topic or course, while done casually or incoherently it can deepen confusion. To further confound matters, there is no universal standard for notation or its use, so what little a student absorbs in one course might not be as portable as he or she would like. I address all of this with my proposed standard, but first address the problems in more detail.

Most authors and instructors use the connectives $\sim$, $\wedge$, $\vee$ or minor variations similarly. However, some then use $\implies$ and $\iff$ where others use $\rightarrow$ and $\leftrightarrow$, and vice-versa. To indicate logical equivalence of compound statements, some authors use the symbol $\equiv$ (used elsewhere to indicate function identities or equality by definition), while others use $\leftrightarrow$, $\iff$, or "iff" meaning "if and only if." (Other notations can also be found but are less common.) In lecture notes, it is not unheard of to see $\implies$ when $\iff$ is also correct and therefore more precise (or even intended), and when it would be very instructive to the students to see the differences between the two; similarly if

$\leftrightarrow$ and $\rightarrow$ are used instead. Indeed, often in some computational argument a string of equivalence is broken, as in $f(x) = x \cdot x \iff f(x) = x^2 \implies f'(x) = 2x$, where of course we can not replace $\implies$ with $\iff$. Of course in a calculus context it seems inadvisable to use the symbol $\rightarrow$ in-line because of its ubiquitous use in limit arguments. Even when we restrict the discussion to the use of logic symbols and their meaning, we have contexts where $P \rightarrow Q$ is supposed to be understood to be a tautology (always true), and others where it is meant to have truth values that may be T or F, depending upon the context. Similarly with $P \leftrightarrow Q$.

The simple notational proposal put forward here is this:

> *That we use $\rightarrow$ and $\leftrightarrow$ as connectives, i.e., logical operations, which can return either truth values* T *or* F*, and $\implies$ and $\iff$ only when the corresponding connectives return tautologies.*

The connectives will be referred to here as "single-line arrows" and the other symbols as "double-line arrows." Thus one may write

$$P \leftrightarrow Q \iff (P \rightarrow Q) \wedge (Q \rightarrow P), \tag{1}$$

$$P \leftrightarrow Q \implies P \rightarrow Q. \tag{2}$$

While the first of these (1) would normally be proved using truth tables by showing that $P \leftrightarrow Q$ and $(P \rightarrow Q) \wedge (Q \rightarrow P)$ have the same truth values for each of the $2^2 = 4$ possible combinations of truth values of the P and Q, it should be noted that it can also be proved by showing that $[P \leftrightarrow Q] \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$ is a tautology.

P	Q	$P \leftrightarrow Q$	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q)$ $\wedge(Q \rightarrow P)$	$[P \leftrightarrow Q]$ $\leftrightarrow [(P \rightarrow Q)$ $\wedge(Q \rightarrow P)]$
T	T	T	T	T	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

Of course it is easier, and time saving, to simply note that the third and sixth columns of the truth table have the same values, and spare the effort of producing the final column. However it is worth noting that showing two compound statements have the same truth values in all cases is equivalent to showing that the connective $\leftrightarrow$ operating on the compound statements will always return T.

To rephrase, the spirit of the longer method of showing equivalence is the notion that $\iff$ means that if its place is taken by $\leftrightarrow$ we get a tautology, after adding necessary grouping symbols. This can be extended to implications, in which $\implies$ is taken to mean that, if replaced by $\rightarrow$, we get a tautology. So to prove (2) we construct a truth table as follows:

P	Q	$P \wedge Q$	P	$(P \wedge Q) \rightarrow P$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

Now consider compound statements $\mathcal{P} = \mathcal{P}(P_1, P_2 \ldots, P_n)$ and $\mathcal{Q} = \mathcal{Q}(P_1, P_2 \ldots, P_n)$, where the $P_1, P_2, \ldots, P_n$ are assumed to be independent component statements, with 2^n possible truth value combinations of T and F. When it is appropriate to write $\mathcal{P} \iff \mathcal{Q}$, or $\mathcal{P} \implies \mathcal{Q}$ for compound statements $\mathcal{P}, Q$, I call these, respectively, a *valid equivalence* and a *valid implication.* Otherwise they would be *fallacies*, and perhaps, unlike statements using the connectives $\leftrightarrow, \rightarrow$, we would usually avoid writing the fallacies (just as we avoid writing $2 = 3$). We can use this notation to display valid argument styles in-line. For instance, consider *modus tollens, i.e.,*

$$\begin{array}{c} P \rightarrow Q \\ \sim Q \\ \hline \therefore \sim P \end{array}$$

which becomes $(P \rightarrow Q) \wedge (\sim Q) \implies \sim P$. A check with a truth table construction shows $[(P \rightarrow Q) \wedge (\sim Q)] \rightarrow (\sim P)$ is indeed a tautology. (I have found that using extra "grouping" characters ( )and [ ] expedites the learning of logic by my calculus students, and thus defers the whole discussion of an "order of operations" until they are more comfortable with the notation.)

The symbols $\iff$ and $\implies$ are robust enough to use in many contexts, including algebraic contexts, in which they can very much enhance student understanding of the logical nature of simple mathematics. For example,

$$\begin{aligned} x - \sqrt{x+6} = 0 &\iff x = \sqrt{x+6} \\ &\implies x^2 = x + 6 \\ &\iff x^2 - x - 6 = 0 \\ &\iff (x-3)(x+2) = 0 \\ &\iff (x = 3) \vee (x = -2). \end{aligned} \tag{3}$$

Students witness where we "broke" our equivalence, allowing us (at this point) only to conclude that $x - \sqrt{x+6} = 0 \implies (x = 3) \vee (x = -2)$, and thus we have a need to "check" our candidate answers, discovering that $x = 3$ will work in the original while $x = -2$ will not, so we finally conclude $x - \sqrt{x+6} = 0 \iff x = 3$. Of course this kind of thing happens all the time, and in fact we can prove (as above) that $P \implies P \vee Q$, as for example $x - \sqrt{x+6} = 0 \iff x = 3 \implies (x = 3) \vee (x = -2)$.

We can also point to cases in algebra and elsewhere for which there is no need to check solutions (excepting for actual errors), as for instance when we have a polynomial equation that we solve by factoring, or a linear equation we solve by steps which only yield equivalent statements. Knowing when we do not need to check the solutions except for errors can be a useful skill in its own right.

At this point in the discussion it is interesting to show students how, if we have $\mathcal{P} \implies \mathcal{Q}$ and $\mathcal{Q} \implies \mathcal{P}$, then we can write $\mathcal{P} \iff \mathcal{Q}$. (It is reasonable on its face, and can also be shown with creative use of truth tables.) This fact is also often useful, such as when we show how the three "row operations" in Gauss-Jordan elimination produce systems with the same solutions, because the operations are reversible. More specifically, it is easy to show that a solution to the original system is contained in the solution of the transformed system, and that "going backwards" we see the inverse operation of the transformation gives that the solution of the transformed system is contained in the original system's solution. However, like squaring both sides of an equation, differentiation is not quite reversible:

$$f(x) = \sin x \implies f'(x) = \cos x \iff (\exists C \in \mathbb{R})[f(x) = \sin x + C]. \tag{4}$$

Similarly, $x = -5 \implies x^2 = 25 \iff (x = 5) \vee (x = -5)$. This simple example reminds students of two important, often forgotten ideas: that squaring both sides of the equation can lose important information about the signs of the original expressions (the technicality which can be buried in the process, as in our original algebraic example (3)), and that $x^2 = k \iff x = \pm\sqrt{k}$. Students are also interested to note that we do not have the same difficulty when cubing both sides: $x = -3 \iff x^3 = -27 \iff x = -3$, assuming $x \in \mathbb{R}$.

More simple examples where we lose equivalence can be found in solving logarithmic equations, such as the following:

$$\begin{aligned} \log x + \log(x-3) = 1 &\implies \log[x(x-3)] = 1 \\ &\iff x(x-3) = 10 \\ &\iff (x = 5) \vee (x = -2). \end{aligned} \tag{5}$$

A quick check shows $x = -2$ is not a solution while $x = 5$ is a solution. We lost equivalence in combining the logarithms, since when $x = -2$ we have $x, x - 3 < 0$ while $x(x-3) > 0$. In fact, the first equivalence is only valid because $10 > 0$ (for the direction $\Longleftarrow$).

When dealing with quantifiers, it also seems appropriate to keep the single-line-arrow connectives "inside" the quantified statements. For instance, since $P \to Q \iff (\sim P) \vee Q$, we can write

$$(\forall x \in S)(P(x) \to Q(x)) \iff (\forall x \in S)((\sim P(x)) \vee Q(x)).$$

Perhaps less clear at first is the one-way nature of the implication

$$[(\forall x \in S)(P(x))] \vee [(\forall x \in S)(Q(x))] \implies (\forall x \in S)(P(x) \vee Q(x)).$$

Once students learn about negating quantified statements and implications $P \to Q$, the following is relatively easy:

$$\begin{aligned} &\sim [(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[|x-a| < \delta \to |f(x) - f(a)| < \epsilon] \\ &\qquad \iff (\exists \epsilon > 0)(\forall \delta > 0)(\exists x)[(|x-a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon)]. \end{aligned}$$

Of course, quantified statements do not conform to the usual truth table based analysis. Indeed, a quantified statement is either true or false, *i.e.,* a tautology or a contradiction. Hence the symbol $\iff$ is appropriate between "true" quantified statements, or between "false" quantified statements.

It can also be useful to define a symbol to represent a generic tautology, and another for a generic contradiction. For these purposes I define $\mathcal{T}$ and $\mathcal{F}$, respectively. For instance, $P \vee (\sim P) \iff \mathcal{T}$, $P \wedge (\sim P) \iff \mathcal{F}$, $\mathcal{T} \vee Q \iff \mathcal{T}$, $\mathcal{F} \vee Q \iff Q$, and so on. These allow for some interesting computations. For instance, if we wish to reason one level above truth tables, we can use well known equivalences to prove, say, $P \implies P \vee Q$ by showing $P \to (P \vee Q) \iff \mathcal{T}$:

$$\begin{aligned} P \to (P \vee Q) &\iff (\sim P) \vee (P \vee Q) \\ &\iff ((\sim P) \vee P) \vee Q \\ &\iff \mathcal{T} \vee Q \\ &\iff \mathcal{T}. \end{aligned}$$

Of course there are some results which surprise students at first, such as $P \to (\sim P) \iff \sim P$, and $(P \to Q) \vee (Q \to P) \iff \mathcal{T}$. However some are quite reasonable, such as $P \to \mathcal{F} \implies \sim P$, which is perhaps best shown with a truth table, which will have $2 \cdot 1 = 2$ possible truth table combinations to check, since P can have two truth values while $\mathcal{F}$ has only one:

P	$\mathcal{F}$	$P \to \mathcal{F}$	$\sim P$	$(P \to \mathcal{F}) \to (\sim P)$
T	F	F	F	T
F	F	T	T	T

In fact, a quick glance at the truth table reveals that $P \to \mathcal{F} \iff (\sim P)$, another interesting computational result for meditation.

I have already attempted to make my case for the second proposition offered here, which is this:

> *That secondary or early undergraduate mathematics education include a standard and coherent introduction to symbolic logic, to improve the clarity of the material and general discussion, including textbooks, lecture discussions and homework in all under-graduate-level mathematics courses, for students studying engineering and the sciences, if not all fields.*

I have already pointed out how some confusing topics in algebra and calculus can be clarified by the arrows showing the flow of implication, whether it be two way ($\iff$) or one way ($\implies$). In my own calculus courses over the last four years, I have seen enough benefits to justify spending four complete lectures on an outline of symbolic logic notation and its use. The outline I have used for those four lectures is as follows:

1. the operators and truth tables with examples from everyday life;
2. logical equivalence with several examples which are mostly intuitive upon reflection;
3. valid implications and arguments along with symbols for generic tautologies ($\mathcal{T}$) and contradictions ($\mathcal{F}$);
4. and finally a short introduction to quantified statements and their negations, some from mathematics and some from everyday life, such as how to negate "for every man there is a woman who loves him," using quantifiers.

With these and a few other innovations which work well with the notation, much clarity can be gained. For instance, the definitions of continuity and a finite limit at a point are less mysterious after some brief experience with quantifiers:[1]

$$\begin{aligned}
&f \text{ continuous}\\
&\quad \text{at } x = a\\
&\quad\iff (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[\,|x-a| < \delta \to |f(x) - f(a)| < \epsilon],
\end{aligned}$$

$$\begin{aligned}
&\lim_{x\to a} f(x) = L\\
&\quad\iff (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x-a| < \delta \to |f(x) - L| < \epsilon].
\end{aligned}$$

From here students can see how the limit is silent at the point $x = a$ (while continuity is not), and how the part of L in the limit is played by $f(a)$ in the continuity definition. Both definitions cut through the seemingly *ad hoc* nature of the explanations one often needs to offer in order to demonstrate the idea of limits without using ϵ-δ definitions. Many questions can be answered by referring back to the definitions, thus achieving coherence across examples, at least for the case of a finite limit at a point.

In fact I have used symbolic logic in ϵ-δ proofs for continuity at points with some success in Calculus 1. For instance, suppose I wish my students to prove that the function $f(x) = 5x - 9$ is continuous at the point $x = 2$. The exercise is broken into two parts: the "scratchwork" and the "proof."

<u>Scratch-work</u>: We want $|f(x) - f(a)| < \epsilon$ to follow from our choice of δ. We work backwards from that statement, with $f(x) = 5x - 9$, $a = 2$, and $f(a) = f(2) = 1$.

$$\begin{array}{rrcll}
 & |f(x) - f(a)| & < & \epsilon & \quad\text{(what we need)}\\
\iff & |f(x) - 1| & < & \epsilon & \\
\iff & |5x - 9 - 1| & < & \epsilon & \\
\iff & |5x - 10| & < & \epsilon & \\
\iff & 5|x - 2| & < & \epsilon & \\
\iff & |x - 2| & < & \frac{1}{5}\epsilon & \quad\text{(how to get it).}
\end{array}$$

Now many texts explain that because we have equivalences, we found our δ. But this is not a good strategy if we look ahead to nonlinear functions, next comes a stand-alone proof.

[1] It was recently pointed out to me that we should make some distinction between equivalence by definition and regular logical equivalence. This suggest a possible third standard, that $\equiv$ be only used for definitions, which is similar to its use when we say, for instance, $f(x) \equiv 0$ in other contexts.

$$(f \text{ continuous at } x = a) \equiv (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[\,|x-a| < \delta \to |f(x) - f(a)| < \epsilon],$$

$$\left(\lim_{x\to a} f(x) = L\right) \equiv (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x-a| < \delta \to |f(x) - L| < \epsilon].$$

<u>Proof</u>: For any $\epsilon > 0$ choose $\delta = \frac{1}{5}\epsilon$. Then $\delta > 0$ exists and satisfies

$$\begin{aligned} |x-2| < \delta \implies & |f(x) - f(2)| = |(5x-9) - 1| \\ & = |5x - 10| = 5|x-2| < 5\delta = 5 \cdot \frac{1}{5}\epsilon = \epsilon, \text{ q.e.d.} \end{aligned}$$

Suppose instead we wish to prove that $f(x) = x^2$ is continuous at $x = 4$. We can attempt the same strategy but it quickly falls short, until we remember which direction of " $\iff$ " we actually need.

<u>Scratch-work</u>: Here $a = 4$ and $f(a) = 16$. We therefore want to choose $\delta > 0$ such that

$$|x-4| < \delta \implies |f(x) - 16| < \epsilon.$$

Working backwards as before we get

$$\begin{aligned} & |f(x) - 16| < \epsilon \\ \iff & |x^2 - 16| < \epsilon \\ \iff & |x+4||x-4| < \epsilon. \end{aligned}$$

Of course $|x+4|$ is not constant, so we are stuck until we realize the direction we really need is $\Longleftarrow$ in the above i.e., $|x-2| < \delta \implies |f(x) - 4| < \epsilon$. We can also have statements before and after our antecedent if the arrows continue to flow the correct direction. Of course we accomplish our implication by employing some *a priori* assumption that $\delta \leq 1$, for instance. Then

$$\begin{aligned} |x-4| < \delta \implies |x-4| < 1 & \implies x \in (3,5) \\ & \implies x + 4 \in (7,9) \\ & \implies |x+4| < 9 \\ & \implies |x+4||x-4| < 9|x-4|. \end{aligned}$$

If this last quantity is less than ϵ when $|x-4| < \delta$, we will have our proof, and that can happen if $9|x-4| < 9\delta \leq \epsilon$, and the last part is true if $\delta \leq \epsilon/9$. We note that we keep thinking, "that can happen *if*," and so we are in fact working backwards again from $|f(x) - f(a)| < \epsilon$, except (1) we do not have equivalence, but our "arrows" are pointing towards $|f(x) - f(a)| < \epsilon$, and (2) we in fact require two things to be present to accomplish $|f(x) - f(a)| < \epsilon$. Indeed, we need δ to accomplish both, $\delta \leq 1$ and $\delta \leq \epsilon/9$, and we do this by taking $\delta = \min\{1, \epsilon/9\}$. From there we have our proof.

<u>Proof</u>: For $\epsilon > 0$, choose $\delta = \min\left\{1, \frac{1}{9}\epsilon\right\}$. Then $\delta > 0$ exists and satisfies

$$|x-4| < \delta \implies |f(x) - 16| = |x+4||x-4| < 9|x-4| < 9\delta \leq 9 \cdot \frac{\epsilon}{9} = \epsilon, \text{ q.e.d.}$$

The level of sophistication required of calculus students for understanding such a proof, and moreover reproducing such a proof, is unremarkable when put so precisely and concisely with the aid of symbolic logic. That tone of sophistication can be sorely missed in later topics if ϵ-δ definitions and proofs are omitted. This is one context in which the symbolic logic can arguably enhance the sophistication of our students.

However, there are pitfalls to avoid when introducing symbolic logic into, say, a first-semester engineering calculus course. First, it should be done with some completeness; a piecemeal introduction, particularly "on the fly," will cause more confusion with most students than it is worth. Second, while most students will enthusiastically embrace symbolic logic, they will also need some correction, as the instructor reads " $\implies$ " where what is meant is "=," for instance, so some patience will be required, as always, but in that sense symbolic logic is no different from other topics. Third, some theorems and arguments instructors take for granted can have surprisingly sophisticated logical structure when forced into symbolic form. While students can learn to produce an ϵ-δ proof of continuity for a linear case reasonably quickly (though not immediately), it becomes much more interesting for the nonlinear cases, where δ must be restricted *a priori*. Part of the solution is to keep the "scratch-work" seeking $\delta = \delta(\epsilon)$ and the actual ϵ-δ proof of continuity separate, which is less necessary but still a good practice for the linear case. Fourth, some rewriting of lecture notes, and some reshuffling of order and priorities may be required to take full advantage of the notation. For instance, I have had reasonable success by introducing the more intuitive topic of continuity before the less intuitive limits, requiring some ϵ-δ proofs of continuity at points (the hardest part of the entire semester for some), then defining continuity on intervals and giving the usual theorems there (including the Intermediate Value Theorem and using it to solve inequalities), and then "breaking" the continuity and introducing limits to describe the behavior of discontinuous functions, as well as the continuous ones, but mostly excluding ϵ-δ proofs for limits in the exercises. Finally, some time is lost initially by including a primer on logic, so some consolidation may be required. For instance, I introduce some easily grasped calculus facts in earlier contexts, for instance the fact that $f' > 0$ on an interval implies f is (strictly) increasing on the interval while $f' < 0$ implies decreasing, and use these for some simple graphing problems long before the Mean Value Theorem is introduced to prove them. In other words, introducing logic does not have to turn calculus into rigorous analysis. An instructor can judge the level of rigor appropriate for individual classes.

In the end, an introduction to logic symbols and their manipulations can help students to clarify their own thinking, help illuminate the flow of a mathematical argument, and better illustrate the idea that mathematics is a language of sorts. This is assuming the logic is introduced in a somewhat complete and coherent fashion. It would also help if it were "portable," i.e., standardized, as well. Simply using $\leftrightarrow$ and $\rightarrow$ as connectives, and $\iff$ and $\implies$ when the respective connectives would return tautologies, and between quantified statements where appropriate, can do a lot toward standardizing the classroom-level use of symbolic logic, with the potential to make symbolic logic a powerful tool for mathematics instruction.

One criticism of this approach is that we can not expect students to perform the calculations better if they are constantly worrying if they should connect them with $\iff$ or $\implies$. This criticism has some merit. A possible relaxation of the approach would be to use $\implies$ throughout the process, until the problem is more or less complete, and then to step back and notice where we in fact have $\iff$. Even if $\iff$ is never used in a particular classroom, $\implies$ can give some road map to the given argument. When useful, the standard presented here gives the instructor some structure to fall back upon.