TWO GENERATED LIE ALGEBRA L WHERE L″≠0 AND DIM(L′/L″)=3

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In the 2003 paper by Csaba Schneider in the Journal of Algebra, the structure of finite *p*-groups G, such that G″≠1 and |G′/G″|=p^3 were found. In the two generator case, Schneider used Lie algebra calculations to inspire the ideas behind the group structure and then extended the group structure to include the cases of more than two generators. In this paper, we begin by examining a few examples and then looking at the structure of the analogous two generated Lie algebra problem from Schneider's paper where $L''\neq 0$ and $\dim(L'/L'')=3$ and comparing this to the examples we created.

To begin, we will provide some background definitions and terminology to be used throughout the paper. For nilpotent Lie algebras, there exists a lower central series $L \supseteq L^2 \supseteq L^3 \supseteq \cdots \supseteq L^{t+1}=0$, where $L^2=[L,L], L^3=[L,L^2],\ldots,L^n=[L,L^{n-1}]$ and L is said to be nilpotent of class *t*. We will define $L^1=\{x\in L \mid [x,L^2]\in L^4\}$, $L'=[L,L]$, and $L''=[L',L']$, so we have $L\supseteq L^1 \supset L'=L^2 \supseteq L^3 \supseteq L^4 \supseteq L^5 \supseteq L''\cdots \supseteq L^{t+1}=0$. For simplicity, we will use the notation $[\cdots[[x_1,x_2],x_3],\cdots x_n]=[x_1,x_2,x_3,\cdots,x_n]$.

For the analogous Lie algebra problem, we will be considering Lie algebras L such that $L''\neq 0$ and $\dim(L'/L'')=3$ and we will also assume that the characteristic of the field of L is not 2. First, it is important to note that if $\dim(L'/L'')=3$ and L has class 5, then $L^6=0$ and there is an one-dimensional jump between each of the L^i's. Furthermore, the $\dim(L/L^1)=1$ so we have $L=\langle a\rangle+L^1$, $L^1=\langle b\rangle+L^2$ and $L^2=\langle [a,b]\rangle+L^3$ since we have a three dimensional jump from L' to L'' and there would be no more dimensions if $L^6=0$.

To coincide with the type of structure we are looking at, we will consider the set of strictly upper triangular matrices $T(n,F)=\{A_{nxn} \mid A$ is strictly upper triangular$\}$. Examining the lower central series of *T*, we see that $T^n=[0]_{nxn}$, so *T* is nilpotent of class n-1. More specifically, *T*(6,F) is also a nilpotent Lie algebra of class 5 and any subalgebra will also be nilpotent.

Consider the Lie algebra $T(6,F)=\{A_{6x6} \mid A$ is strictly upper triangular$\}$ and let L be the Lie algebra generated by A=$E_{12}+E_{34}+E_{45}$ and B= $E_{23}+E_{56}$, where E_{ij} is the matrix of all 0's except a 1 in the (i,j) position. We generate the algebra by bracketing all possible combinations and we get the following: [A,B]= $E_{13}+E_{46}-E_{24}$, [A,B,A]= $2E_{14}-E_{25}-E_{36}$, [A,B,A,A]=$3E_{15}$, [A,B,A,A,A]=0, [A,B,B]=0, [A,B,A,B]=0, [A,B,A,A,B]=$3E_{16}$, and [[A,B,A],[A,B]]= $3E_{16}$. So we construct a table to show the structure of L:

	A	B	[A,B]	[A,B,A]	[A,B,A,A]	[A,B,A,A,B]
A	0	[A,B]	-[A,B,A]	-[A,B,A,A]	0	0
B	-[A,B]	0	0	0	-[A,B,A,A,B]	0
[A,B]	[A,B,A]	0	0	-[A,B,A,A,B]	0	0
[A,B,A]	[A,B,A,A]	0	[A,B,A,A,B]	0	0	0
[A,B,A,A]	0	[A,B,A,A,B]	0		0	0
[A,B,A,A,B]	0	0	0	0	0	0

Table 1: Structure for Example 1

So this Lie algebra *L* is generated by two elements, A and B, the dimension of *L* is 6, where the basis for *L*={A, B, [A,B], [A,B,A], [A,B,A,A], [A,B,A,A,B]}. Also, we have *L′*=[L,L]={[A,B], [A,B,A], [A,B,A,A], [A,B,A,A,B]} so the dimension of *L′* is 4 and *L″*=[*L′*, *L′*] = {[A,B,A,A,B]} so the dimension of *L″* is 1. Hence, dim(*L′*/*L″*)=3 and *L″≠0*. Notice *L* is nilpotent of class 5 and that the center of *L*=Z(*L*)={[A,B,A,A,B]}, so it is also one dimensional.

If we take a look at a similar example, using the Lie algebra $T(6,F)=\{A_{6x6} \mid A$ is strictly upper triangular} but now let *L* be the Lie algebra generated by A=E_{12}+E_{34}+E_{45} and B= E_{23}+E_{56}+3E_{35}. Again, we use the bracket structure to generate the algebra: [A,B]= E_{13}+E_{46}-E_{24} , [A,B,A]= 2E_{14}-E_{25}-E_{36}, [A,B,A,A]=3E_{15}, [A,B,A,A,A]=0, [A,B,B]=3E_{15}, [A,B,B,A]=0, [A,B,B,B]=3E_{16}, [A,B,A,B]=0, [A,B,A,A,B]= 3E_{16} , and [[A,B,A],[A,B]]= 3E_{16} . In this case we get the following table:

	A	B	[A,B]	[A,B,A]	[A,B,A,A]	[A,B,A,A,B]
A	0	[A,B]	-[A,B,A]	-[A,B,A,A]	0	0
B	-[A,B]	0	-[A,B,A,A]	0	-[A,B,A,A,B]	0
[A,B]	[A,B,A]	[A,B,A,A]	0	-[A,B,A,A,B]	0	0
[A,B,A]	[A,B,A,A]	0	[A,B,A,A,B]	0	0	0
[A,B,A,A]	0	[A,B,A,A,B]	0		0	0
[A,B,A,A,B]	0	0	0	0	0	0

Table 2: Structure for Example 2

So this Lie algebra is also generated by two elements, A and B, the dimension of *L* is 6, where the basis for is the same as the previous example. The dimension of *L′* is 4 and *L″*=[*L′*, *L′*] = {[A,B,A,A,B]} so the dimension of *L″* is 1. Hence, dim(*L′*/*L″*)=3 and *L″≠0*. The algebra is also nilpotent of class 5 and the center is one dimensional as well. The only difference in these two examples is that [A,B,B]=[A,B,A,A] instead of 0 in the second example. We can see that both of these examples are of a two generated Lie algebra *L*, where *L″*≠0 and dim(*L′*/*L″*)=3.

From Schneider's paper, we were able to use his group theory ideas to examine the analogous Lie algebra corollaries and theorems as well as some additional theorems to aid in the construction of these specific Lie algebras. The following are the most useful theorems we have proved that were used to examine the structure of *L*. The first theorem is an analogous theorem from Phillip Hall and the second is one from Csaba Schneider.

Theorem: *Suppose L is a nilpotent Lie algebra with characteristic not 2 such that dim(L′/L″)=3 and dim(L″) ≥ 1. Then the dim(L″) =1.*

Theorem: *Let L be a 2-generated finite dimensional nilpotent Lie algebra such that dim(L′/L″)=3 and L″≠0. Then generators a and b of L can be chosen such that the following hold:*

(i) $L^2 = <[a,b]> + L^3$

(ii) $L^3 = <[a,b,a]> + L^4$ *and* $[a,b,b] \in L^4$

(iii) $L^4 = <[a,b,a,a]> + L^5$ *and* $[a,b,a,b] \in L^5$

(iv) $L^5 = <[a,b,a,a,b]> + L^6$ *and* $[a,b,a,a,a] \in L^6$

The above theorems helps give the general structure of this particular Lie algebra *L* with properties *L*=<a,b>, *L*″≠0 and dim(*L*′/*L*″)=3. We also know that $L = <a> + L^1$, $L^1 = <b> + L^2$ and $L^2 = <[a,b]> + L^3$ from before so we can construct a basis table to show the general structure. There are only two multiplications not given by the above theorem which we have found to be [b,[a,b,a]] = -[a,b,a,b] = [a,[a,b,b]] + [[a,b],[a,b]] = -[a,b,b,a]=0 and [a,b,b] is in L^4, so [a,b,b]=e[a,b,a,a]+f[a,b,a,a,b], where e and f are scalars. What follows is the table for generalized Lie algebra *L*:

	a	b	[a,b]	[a,b,a]	[a,b,a,a]	[a,b,a,a,b]
a	0	[a,b]	-[a,b,a]	-[a,b,a,a]	0	0
b	-[a,b]	0	**	0	-[a,b,a,a,b]	0
[a,b]	[a,b,a]	**	0	-[a,b,a,a,b]	0	0
[a,b,a]	[a,b,a,a]	0	[a,b,a,a,b]	0	0	0
[a,b,a,a]	0	[a,b,a,a,b]	0	0	0	0
[a,b,a,a,b]	0	0	0	0	0	0

Table 3: General Structure of *L*
**[a,b,b]=e[a,b,a,a]+f[a,b,a,a,b]

By using a change of basis we were able to get two isomorphism classes of *L*. One with the same structure above plus [a,b,b]=0, and the second with the same structure above and [a,b,b]=[a,b,a,a]. We can see that the general Lie algebra *L* is six dimensional with basis {a, b, [a,b], [a,b,a], [a,b,a,a], [a,b,a,a,b]}, that *L*′=[L,L]={[a,b], [a,b,a], [a,b,a,a], [a,b,a,a,b]} so the dimension of *L*′ is 4 and *L*″=[*L*′, *L*′] = {[a,b,a,a,b]} so the dimension of *L*″ is 1. Hence, dim(*L*′/*L*″)=3 and *L*″≠0, *L* is nilpotent of class 5, and the center *Z*(*L*)={[a,b,a,a,b]} is one dimensional.

We can now see that the two examples created in *T*(6,F) are precisely one example from each of the possible isomorphism classes we were able to find. In fact, we found that these are the only two possible structures for a Lie algebra *L* with these particular properties.