Three Great (But Lesser Known Theorems)
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Sturm's Theorem

From Nathan Jacobson's wonderful book <u>Basic Algebra I (2e)</u>

Jacques Charles François Sturm (September 29, 1803 - December 15, 1855)

The **Standard Sequence** for a polynomial f(x) is
$f_0(x) = f(x)$
$f_1(x) = f'(x)$
$f_0(x) = q_1(x) f_1(x) - f_2(x),$ deg f_2 < deg f_1
$f_1(x) = q_2(x) f_2(x) - f_3(x),$ deg f_3 < deg f_2
. . .
$f_{i-1}(x) = q_i(x) f_i(x) - f_{i+1}(x),$ deg f_{i+1} < deg f_i
. . .
$f_{s-1}(x) = q_s(x) f_s(x),$ that is, $f_{s+1}(x) = 0$

Sturm's Theorem - Let f(x) be a polynomial of positive degree with real coefficients and let {$f_0(x) = f(x), f_1(x) = f'(x), f_2(x), \ldots, f_s(x)$} be the standard sequence for f(x). Assume [a, b] is an interval such that $f(a) \neq 0 \neq f(b)$. Then the number of distinct roots of f(x) in (a, b) is $V_a - V_b$ where V_c denotes the number of variations in sign of {$f_0(c), f_1(c), \ldots, f_s(c)$}.

Note - drop any 0's from the sequence.

> f0:=x^3-7*x+6; f1:=diff(f0,x); f2:=-rem(f0,f1,x); f3:=-rem(f1,f2,x); f4:=-rem(f2,f3,x);

$$f0 := x^3 - 7x + 6$$
$$f1 := 3x^2 - 7$$
$$f2 := -6 + \frac{14}{3}x$$
$$f3 := \frac{100}{49}$$
$$f4 := 0$$

> [subs(x=0,f0),subs(x=0,f1),subs(x=0,f2),subs(x=0,f3)];

$$\left[6, -7, -6, \frac{100}{49} \right]$$

> [subs(x=1.5,f0),subs(x=1.5,f1),subs(x=1.5,f2),subs(x=1.5,f3)];

$$\left[-1.125, -0.25, 1.000000000, \frac{100}{49} \right]$$

> [subs(x=3,f0),subs(x=3,f1),subs(x=3,f2),subs(x=3,f3)];

$$\left[12, 20, 8, \frac{100}{49} \right]$$

> fsolve(f0,x);

$$-3., 1., 2.$$

```
> f0:=x^5-7.8*x^4+22.36*x^3-29.16*x^2+16.96*x-3.36;
  f1:=diff(f0,x); f2:=-rem(f0,f1,x); f3:=-rem(f1,f2,x);
  f4:=-rem(f2,f3,x);f5:=-rem(f3,f4,x); f6:=-rem(f4,f5,x);
```

$$f0 := x^5 - 7.8\,x^4 - 3.36 + 22.36\,x^3 + 16.96\,x - 29.16\,x^2$$

$$f1 := 5\,x^4 - 31.2\,x^3 + 67.08\,x^2 + 16.96 - 58.32\,x$$

$$f2 := -1.931520000 + 4.627840000\,x - 3.432960000\,x^2 + 0.7904000000\,x^3$$

$$f3 := 6.214820764 - 9.424511138\,x + 3.384746513\,x^2$$

$$f4 := -0.3308898615 + 0.2542821669\,x$$

$$f5 := 0.3176096024$$

$$f6 := 0$$

```
> [subs(x=0,f0),subs(x=0,f1),subs(x=0,f2),subs(x=0,f3),subs(x=0
  ,f4),subs(x=0,f5)];
```

$$[-3.36, 16.96, -1.931520000, 6.214820764, -0.3308898615, 0.3176096024]$$

```
> [subs(x=1.5,f0),subs(x=1.5,f1),subs(x=1.5,f2),subs(x=1.5,f3),
  subs(x=1.5,f4),subs(x=1.5,f5)];
```

$$[0.04125, 0.4225, -0.046320000, -0.306266292, 0.0505333889, 0.3176096024]$$

```
> [subs(x=3,f0),subs(x=3,f1),subs(x=3,f2),subs(x=3,f3),subs(x=3
  ,f4),subs(x=3,f5)];
```

$$[0., 8.32, 2.39616000, 8.40400597, 0.4319566392, 0.3176096024]$$

```
> solve(f0,x);
```

$$1., 2., 3., 0.4000000000, 1.400000000$$

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>
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Sarkovskii's Theorem

From Robert L. Devaney's wonderful book <u>An Introduction to Chaotic Dynamical Systems (2e)</u>

Background
$f^1 = f$
$f^n = f \circ f^{n-1}$
The point, x, is a <u>fixed point</u> of f if f(x) = x.
The point, x, is a <u>periodic point of period n for f</u> if x is a fixed point of f^n and is not a fixed point for f^k for any k < n.

Sarkovskii's Ordering

$3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \ldots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \ldots \rhd 2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$

Theorem - Suppose f: **R** → **R** is continuous. Suppose f has a periodic point of period k. If k $\rhd$ n in the Sarkovskii ordering, then f also has a periodic point of period n.

Sarkovskii's Theorem first appeared in the Ukrainian Maths Journal in 1964 (Vol 16, pp 61-71).

In December 1975, Tien-Yien Li and James A. Yorke published a paper in the *American Mathematical Monthly* (**82**, 985-992.) entitled "Period Three Implies Chaos." This paper proved a similar result on intervals. They didn't mention Sarkovskii. It is possible they were unaware of his result.

The converse is also true. There are maps which have periodic points of period k and no "higher" period according to the Sarkovskii ordering.

For example, consider the following map.
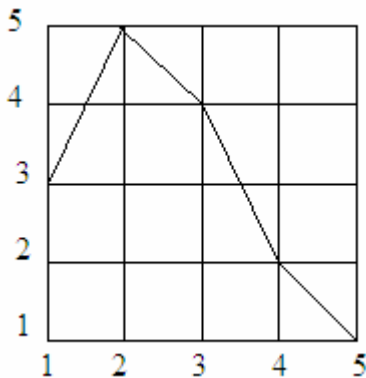
Let f(x) = 3 for x < 1 and f(x) = 1 for x > 5. $f^5(1) = f^4(3) = f^3(4) = f^2(2) = f(5) = 1$. Thus, 1, 2, 3, 4 and 5 are all period 5 points. Note that $f^3[1, 2] = [2, 5]$, $f^3[2, 3] = [3, 5]$ and $f^3[4, 5] = [1, 4]$ so the only possible period 3 points are in [3, 4] since $f^3[3, 4] = [1, 5]$. We see

f[3, 4] above. It contains a fixed point, x = 10/3. But, note from the graph below of f^3 that f^3 is monotonically decreasing on [3, 4] so the only fixed point of f^3 is 10/3, the fixed point of f.
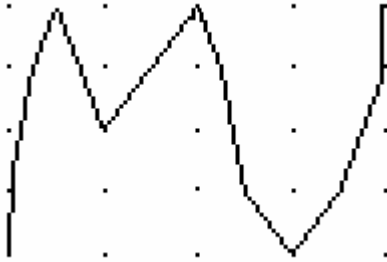
No point outside of [1, 5] will be periodic since all iterations of them will be in [1, 5]. Thus we have a period 5 point but no period 3 point.

Similar functions can be constructed having period 2n+1 points but no 2n-1 points for n > 2.

For the even periods, the construction is a little messier and I'll refer you to Devaney's book.

Solution by radicals of 3rd and 4th degree polynomials

(These formulas come from the wonderful book <u>An Introduction to the Theory of Groups</u> by Joseph Rotman.)

Cubic Formula

Consider $x^3 + Ax^2 + Bx + C$.

Let $X = x + A/3$. Then

$$x^3 + Ax^2 + Bx + C = \left(X - \frac{A}{3}\right)^3 + A\left(X - \frac{A}{3}\right)^2 + B\left(X - \frac{A}{3}\right) + C$$

$$= X^3 - AX^2 + \frac{XA^2}{3} - \frac{A^3}{27} + AX^2 - \frac{2XA^2}{3} + \frac{A^3}{9} + BX - \frac{AB}{3} + C$$

$$= X^3 + X^2(-A + A) + X\left(\frac{A^2}{3} - \frac{2A^2}{3} + B\right) + \left(\frac{-A^3}{27} + \frac{A^3}{9} - \frac{AB}{3} + C\right)$$

$$= X^3 + 0\,X^2 + Q\,X + R$$

$$= X^3 + Q\,X + R$$

where $Q = \dfrac{A^2}{3} - \dfrac{2A^2}{3} + B$ and $R = \dfrac{-A^3}{27} + \dfrac{A^3}{9} - \dfrac{AB}{3} + C$.

Suppose $a^3 + Qa + R = 0$.

Write $a = b + c$ where b and c are to be determined.

Then $a^3 = (b + c)^3 = b^3 + c^3 + 3(cb^2 + bc^2) = b^3 + c^3 + 3abc$.

Then $b^3 + c^3 + (3bc + Q)a + R = 0$. (*)

Also suppose $bc = -Q/3$. Then the middle term in (*) vanishes.

Therefore (*) becomes $b^3 + c^3 = -R$ and $b^3c^3 = -Q^3/27$. Thus, from substitution we have

$$b^3 - \frac{Q^3}{27b^3} = -R,$$

or

$$\left(b^3\right)^2 + R\,b^3 - \frac{Q^3}{27} = 0.$$

By the quadratic formula we have

$$b^3 = \frac{-R \pm \sqrt{R^2 + \dfrac{4Q^3}{27}}}{2}$$

and, since $b^3 + c^3 = -R$,

$$c^3 = \frac{-R \mp \sqrt{R^2 + \dfrac{4Q^3}{27}}}{2}.$$

Thus

$$a = b + c = \left(\frac{-R}{2} + \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}}\right)^{1/3} + \left(\frac{-R}{2} - \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}}\right)^{1/3}$$

where the cube roots are chosen so the product bc = -Q/3. This is a root of the altered cubic equation so x = a - A/3. Once that root is divided out you have a quadratic which can be solved easily.

$x^3 + Ax^2 + Bx + C.$
$x = a - A/3$

$$= \left(\frac{-R}{2} + \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}}\right)^{1/3} + \left(\frac{-R}{2} - \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}}\right)^{1/3} - \frac{A}{3}$$

$$= \left(\frac{-\left(\dfrac{2A^3}{27} - \dfrac{AB}{3} + C\right)}{2} + \sqrt{\frac{\left(\dfrac{2A^3}{27} - \dfrac{AB}{3} + C\right)^2}{4} + \frac{\left(B - \dfrac{A^2}{3}\right)^3}{27}}\right)^{1/3}$$

$$+ \left(\frac{-\left(\dfrac{2A^3}{27} - \dfrac{AB}{3} + C\right)}{2} - \sqrt{\frac{\left(\dfrac{2A^3}{27} - \dfrac{AB}{3} + C\right)^2}{4} + \frac{\left(B - \dfrac{A^2}{3}\right)^3}{27}}\right)^{1/3} - \frac{A}{3}$$

$$= \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C + \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3}\right)^{(1/3)}$$

$$+ \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C - \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3}\right)^{(1/3)} - \frac{1}{3}A$$

$x^3 + Ax^2 + Bx + C = 0$ has solution

$$a = \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C + \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3}\right)^{(1/3)}$$

$$+ \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C - \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3}\right)^{(1/3)} - \frac{1}{3}A$$

and $x^3 + Ax^2 + Bx + C = (x - a)(x^2 + bx + c)$.

Quartic Formula

For the quartic $x^4 + Ax^3 + Bx^2 + Cx + D$, let $x = X - A/4$. This gives

$$x^4 + A x^3 + B x^2 + C x + D$$

$$= \left(X - \frac{A}{4}\right)^4 + A\left(X - \frac{A}{4}\right)^3 + B\left(X - \frac{A}{4}\right)^2 + C\left(X - \frac{A}{4}\right) + D$$

$$= X^4 - A X^3 + \frac{3 A^2 X^2}{8} - \frac{X A^3}{16} + \frac{A^4}{256} + A X^3 - \frac{3 X^2 A^2}{4}$$

$$+ \frac{3 X A^3}{16} - \frac{A^4}{64} + B X^2 - \frac{B A X}{2} + \frac{B A^2}{16} + CX - \frac{A C}{4} + D$$

$$= X^4 + X^3 (-A + A) + X^2 \left(\frac{3 A^2}{8} - \frac{3 A^2}{4} + B\right)$$

$$+ X \left(\frac{-A^3}{16} + \frac{3 A^3}{16} - \frac{B A}{2} + C\right) + \left(\frac{A^4}{256} - \frac{A^4}{64} + \frac{B A^2}{16} - \frac{A C}{4} + D\right)$$

$$= X^4 + QX^2 + RX + S$$

where
$$Q = \frac{3 A^2}{8} - \frac{3 A^2}{4} + B, \quad R = \frac{-A^3}{16} + \frac{3 A^3}{16} - \frac{B A}{2} + C, \quad S = \frac{A^4}{256} - \frac{A^4}{64} + \frac{B A^2}{16} - \frac{A C}{4} + D.$$

$X^4 + QX^2 + RX + S = (X^2 + KX + L)(X^2 - KX + M)$ where finding K, L and M reduces us to two quadratics.

Expanding the right hand side and equating coefficients of like powers of x gives $L + M - K^2 = Q$, $K(M - L) = R$ and $LM = S$.

Thus, from the first two of these we get
$2M = K^2 + Q + R/K$ and $2L = K^2 + Q - R/K$. (**)

Substituting these into the third equation gives $(K^3 + QK + R)(K^3 + QK - R) = 4SK^2$ which becomes $K^6 + 2QK^4 + (Q^2 - 4S)K^2 - R^2 = 0$ which is a cubic in K^2 and can thus be solved using the technique given above.

We then can solve for L and M in (**) giving two quadratics which we can solve using the quadratic formula.

One solution is:

$x1 :=$

$-\dfrac{1}{12}$ sqrt(6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 + 12 sqrt(-54 A^3 B C d + 240 A B^2 C d + 12 B^3 d A^2 - 3 A^2 B^2 C^2

$\qquad$ - 54 A B C^3 - 432 d^2 A^2 B + 18 d A^2 C^2 - 432 d B C^2 - 48 B^4 d + 12 B^3 C^2 + 81 d^2 A^4 + 384 d^2 B^2 + 12 C^3 A^3 + 81 C^4

$\qquad$ - 768 d^3 + 576 C A d^2))^(1/3) + 6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 - 4 sqrt(59049 - 4 B^6 + 36 B^4 A C

$\qquad$ - 144 B^4 d - 108 A^2 B^2 C^2 + 864 A B^2 C d - 1728 d^2 B^2 + 108 C^3 A^3 - 1296 d A^2 C^2 + 5184 C A d^2 - 6912 d^3))^(1/3)

$\qquad$ + 9 A^2 - 24 B) + $\dfrac{1}{12}$ sqrt$\Bigg(6$ (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 + 12 sqrt(-54 A^3 B C d + 240 A B^2 C d

$\qquad$ + 12 B^3 d A^2 - 3 A^2 B^2 C^2 - 54 A B C^3 - 432 d^2 A^2 B + 18 d A^2 C^2 - 432 d B C^2 - 48 B^4 d + 12 B^3 C^2 + 81 d^2 A^4

$\qquad$ + 384 d^2 B^2 + 12 C^3 A^3 + 81 C^4 - 768 d^3 + 576 C A d^2))^(1/3) + 6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 - 4 sqrt(

$\qquad$ 59049 - 4 B^6 + 36 B^4 A C - 144 B^4 d - 108 A^2 B^2 C^2 + 864 A B^2 C d - 1728 d^2 B^2 + 108 C^3 A^3 - 1296 d A^2 C^2

$\qquad$ + 5184 C A d^2 - 6912 d^3))^(1/3) + 9 A^2 - 24 B + $\Bigg(54 \Bigg(A^3$ - 4 A B + 8 C + $\dfrac{1}{2} A^2$ sqrt(6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B

$\qquad$ + 108 C^2 + 12 sqrt(-54 A^3 B C d + 240 A B^2 C d + 12 B^3 d A^2 - 3 A^2 B^2 C^2 - 54 A B C^3 - 432 d^2 A^2 B + 18 d A^2 C^2

$\qquad$ - 432 d B C^2 - 48 B^4 d + 12 B^3 C^2 + 81 d^2 A^4 + 384 d^2 B^2 + 12 C^3 A^3 + 81 C^4 - 768 d^3 + 576 C A d^2))^(1/3) + 6 (8 B^3

$\qquad$ - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 - 4 sqrt(59049 - 4 B^6 + 36 B^4 A C - 144 B^4 d - 108 A^2 B^2 C^2 + 864 A B^2 C d

$\qquad$ - 1728 d^2 B^2 + 108 C^3 A^3 - 1296 d A^2 C^2 + 5184 C A d^2 - 6912 d^3))^(1/3) + 9 A^2 - 24 B) - $\dfrac{4}{3} B$ sqrt(6 (8 B^3 - 36 A B C

$\qquad$ + 108 d A^2 - 288 d B + 108 C^2 + 12 sqrt(-54 A^3 B C d + 240 A B^2 C d + 12 B^3 d A^2 - 3 A^2 B^2 C^2 - 54 A B C^3 - 432 d^2 A^2 B

$\qquad$ + 18 d A^2 C^2 - 432 d B C^2 - 48 B^4 d + 12 B^3 C^2 + 81 d^2 A^4 + 384 d^2 B^2 + 12 C^3 A^3 + 81 C^4 - 768 d^3 + 576 C A d^2))^(1/3) +

$\qquad$ 6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 - 4 sqrt(59049 - 4 B^6 + 36 B^4 A C - 144 B^4 d - 108 A^2 B^2 C^2

$\qquad$ + 864 A B^2 C d - 1728 d^2 B^2 + 108 C^3 A^3 - 1296 d A^2 C^2 + 5184 C A d^2 - 6912 d^3))^(1/3) + 9 A^2 - 24 B) - $\dfrac{1}{27}$ (6 (8 B^3

$\qquad$ - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 + 12 sqrt(-54 A^3 B C d + 240 A B^2 C d + 12 B^3 d A^2 - 3 A^2 B^2 C^2 - 54 A B C^3

$\qquad$ - 432 d^2 A^2 B + 18 d A^2 C^2 - 432 d B C^2 - 48 B^4 d + 12 B^3 C^2 + 81 d^2 A^4 + 384 d^2 B^2 + 12 C^3 A^3 + 81 C^4 - 768 d^3

$\qquad$ + 576 C A d^2))^(1/3) + 6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 - 4 sqrt(59049 - 4 B^6 + 36 B^4 A C - 144 B^4 d

$\qquad$ - 108 A^2 B^2 C^2 + 864 A B^2 C d - 1728 d^2 B^2 + 108 C^3 A^3 - 1296 d A^2 C^2 + 5184 C A d^2 - 6912 d^3))^(1/3) + 9 A^2 - 24 B)^(3/2

$\qquad\Big)\Bigg)\Bigg) \Big/ $ (sqrt(6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 + 12 sqrt(-54 A^3 B C d + 240 A B^2 C d + 12 B^3 d A^2

$\qquad$ - 3 A^2 B^2 C^2 - 54 A B C^3 - 432 d^2 A^2 B + 18 d A^2 C^2 - 432 d B C^2 - 48 B^4 d + 12 B^3 C^2 + 81 d^2 A^4 + 384 d^2 B^2

$\qquad$ + 12 C^3 A^3 + 81 C^4 - 768 d^3 + 576 C A d^2))^(1/3) + 6 (8 B^3 - 36 A B C + 108 d A^2 - 288 d B + 108 C^2 - 4 sqrt(59049 - 4 B^6

$\qquad$ + 36 B^4 A C - 144 B^4 d - 108 A^2 B^2 C^2 + 864 A B^2 C d - 1728 d^2 B^2 + 108 C^3 A^3 - 1296 d A^2 C^2 + 5184 C A d^2 - 6912 d^3)

$\qquad$)^(1/3) + 9 A^2 - 24 B))$\Bigg) - \dfrac{1}{4} A$

The other solutions are similar.