

Pursuit Curves

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Abstract

The classic pursuit curve from differential equations will be derived, and then variations will be explored using Maple.

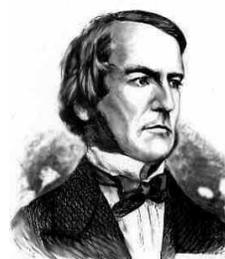
Definition

The idea of a pursuit curve is that a point, which we will call the rabbit, follows a prescribed curve. The rabbit is followed by another point, which we will call the fox. Two conditions will be specified to determine a pursuit curve:

1. The fox heads directly towards the rabbit.
2. The fox's speed is directly proportional to the rabbit's.



Pierre Bouguer



George Boole

History

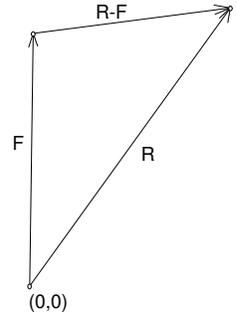
Pursuit curves were considered in general by the French scientist Pierre Bouguer in 1732. However, the term “pursuit curve” was first defined by George Boole in his “Treatise on differential equations” in 1859. The curved path described by a fighter plane making an attack on a moving target while holding the proper aiming allowance is a pursuit curve, so such curves are relevant to current military research.

Derivation of the General Pursuit Curve

We will only give the derivation for a pursuit curve in the plane, but the derivation in R^n is similar. Let the rabbit's position be given by the parametric function $\mathbf{R}(t)$, and let the fox's position be given by $\mathbf{F}(t) = \langle x(t), y(t) \rangle$. By Property 2 in the definition of a pursuit curve, the ratio of the rabbit's and the fox's speed is a constant, which we will call $k = \frac{\text{fox's speed}}{\text{rabbit's speed}}$.

Property 1 in the definition of a pursuit curve, and the accompanying diagram make it clear that the unit tangent vector to the fox's curve is

$\frac{\mathbf{F}'}{\|\mathbf{F}'\|} = \frac{\mathbf{R} - \mathbf{F}}{\|\mathbf{R} - \mathbf{F}\|}$. By the definition of k , $\|\mathbf{F}'\| = k\|\mathbf{R}'\|$. Substitute this into the previous equation to obtain the vector differential equation describing the general pursuit curve: $\mathbf{F}' = k\|\mathbf{R}'\| \frac{\mathbf{R} - \mathbf{F}}{\|\mathbf{R} - \mathbf{F}\|}$



This system of nonlinear differential equations must be solved by numerical methods except for special cases. It is left to the reader as an exercise to use the chain rule to see that the parameterization of the rabbit's path will not affect the shape of the fox's path. Thus, we may parameterize the rabbit's path using any convenient function, and it does not matter that the rabbit's speed may not be constant.

Special Case of Pursuing a Straight Line Target

Without loss of generality, we may assume that the rabbit runs up the y -axis, and parameterize its path by $\mathbf{R}(t) = \langle 0, rt \rangle$. Let the fox's initial position be given by $\mathbf{F}(0) = \langle c, 0 \rangle$, where c is a positive constant. The vector differential equation for the fox simplifies to the system

$$\begin{cases} x' = -\frac{krx}{\sqrt{x^2 + (rt - y)^2}} \\ y' = \frac{kr(rt - y)}{\sqrt{x^2 + (rt - y)^2}} \end{cases}$$

Obtain $\frac{dy}{dx} = \frac{y'}{x'} = \frac{y - rt}{x}$ from the chain rule and this system. Differentiate both sides with

respect to x and simplify to obtain $x \frac{d^2y}{dx^2} = -r \frac{dt}{dx}$. Use the chain rule and Calculus to obtain

$$\frac{dt}{dx} = \frac{dt}{ds} \cdot \frac{ds}{dx} = -\frac{1}{kr} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \text{ Note that } \frac{ds}{dx} \text{ is negative because } s \text{ increases as } x \text{ decreases.}$$

Eliminate $\frac{dt}{dx}$ from the last two equations to get the second order, nonlinear differential

$$\text{equation } kx \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \text{ We may reduce the order by substituting } v = \frac{dy}{dx} \text{ where } v = 0$$

and $x = c$ to get $k \frac{dv}{dx} = \sqrt{1 + v^2}$. This separable differential equation is solved to obtain

$$\frac{dy}{dx} = \frac{1}{2} \left[\left(\frac{x}{c}\right)^{1/k} - \left(\frac{c}{x}\right)^{1/k} \right]. \text{ The last equation is integrated again to get the fox's path, albeit in}$$

nonparametric form:

$$y = \begin{cases} \frac{1}{2} \left[\frac{x^{1+1/k}}{c^{1/k} (1+1/k)} - \frac{c^{1/k} x^{1-1/k}}{1-1/k} \right] + \frac{ck}{k^2-1} & \text{if } k \neq 1 \\ \frac{1}{2} \left[\frac{x^2 - c^2}{2c} - c \ln \frac{x}{c} \right] & \text{if } k = 1 \end{cases}$$

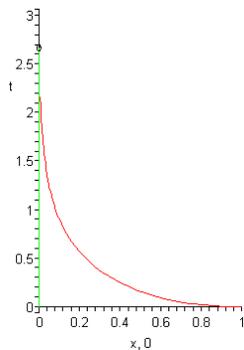
If $k > 1$, which corresponds to the rabbit running faster than the fox, the fox will catch the rabbit when $x = 0$, which implies $y = \frac{ck}{k^2-1}$. The time it will take the fox to catch the rabbit will be $t = \frac{y}{r}$, and the total distance that the fox runs will be $krt = \frac{ck^2}{k^2-1}$.

It is more interesting to consider the second case $k = 1$, which corresponds to the rabbit running at the same speed as the fox. In this case, $\frac{dt}{dx} = -\frac{1}{r} \sqrt{1 + \frac{1}{4} \left(\frac{x-c}{c-x} \right)^2}$, which is integrated to

obtain $t = \frac{c}{2r} \left[\frac{1}{2} - \frac{x^2}{2c^2} - \ln \frac{x}{c} \right]$. Thus, the difference between the y coordinates of the rabbit and

fox is $y_{rabbit} - y_{fox} = rt - \frac{1}{2} \left[\frac{x^2 - c^2}{2c} - c \ln \frac{x}{c} \right] = \frac{c}{2} \left[1 - \left(\frac{x}{c} \right)^2 \right]$. Therefore,

$\lim_{t \rightarrow \infty} \| \mathbf{R}(t) - \mathbf{F}(t) \| = \lim_{x \downarrow 0} (y_{rabbit} - y_{fox}) = \frac{c}{2}$. In other words, the fox will only cut the distance between the rabbit and himself by 2 in the long run. The poor fox will never catch the rabbit!



The computer algebra software Maple dsolve and deplot commands were used to create the following curves. Originally, all the pursuit curves that I created in Maple were animated in a PowerPoint demonstration. Interestingly, the surface obtained by revolving the fox's curve about the y -axis is a model for Lobachevsky's version of non-Euclidean geometry (1829).



Nikolai
Ivanovich
Lobachevsky
(1792-1856)

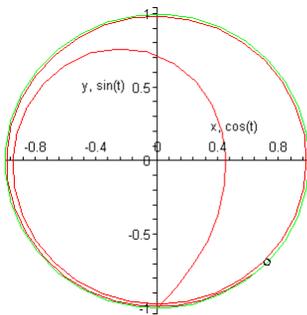
Here are typical Maple commands that were used to create the above and subsequent pursuit curves:

```
> with(plots):
> with(LinearAlgebra):
```

Warning, the name changecoords has been redefined

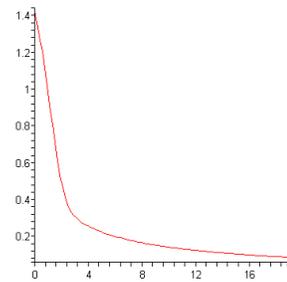
```
> k:=.8:
> R:=<cos(3*t)*cos(t),cos(3*t)*sin(t)>:
> F:=<x(t),y(t)>:
> FP:=map(z->diff(z,t),F):
> RS:=k*(R-F)*Norm(map(z->diff(z,t),R),2,conjugate=false)/Norm(R-F,2,conjugate=false);
> tmax:=10:
> fmax:=200:
> p:= dsolve({FP[1]=RS[1],FP[2]=RS[2],x(0)=0,y(0)=0}, {x(t),y(t)},type=numeric);
> fp:=odeplot(p,[[x(t),y(t)],[R[1],R[2]]],0..tmax,frames=fmax):
> rp:=animate(pointplot,[[[R[1],R[2]]], symbol=circle, symbolsize=10],t=0..tmax, frames=fmax):
> display(fp,rp);
```

Special Case of a the Rabbit Moving in a Circle



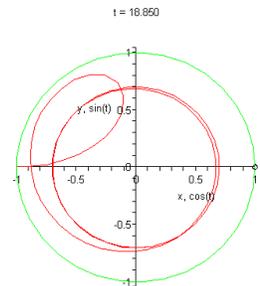
Trajectories

If $k = 1$, note that the distance between the fox and the rabbit is approaching zero. If $k > 1$, then the fox will catch the rabbit in finite time.

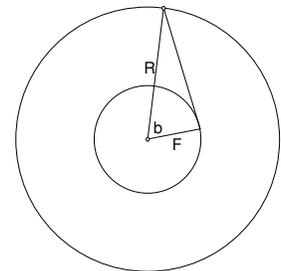


Distance between the Animals vs. Time

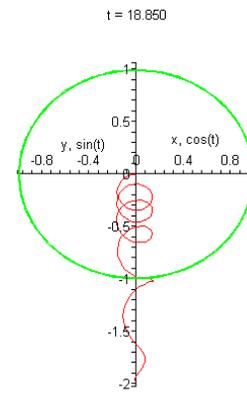
If $k < 1$, then the fox is attracted to a circle with strictly smaller diameter. In the picture shown here, the fox is traveling at 70% the rabbit's speed.



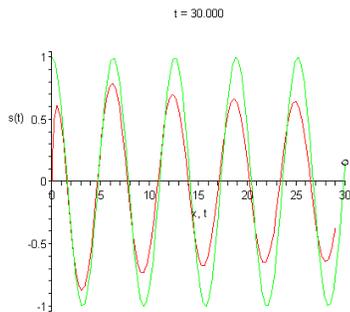
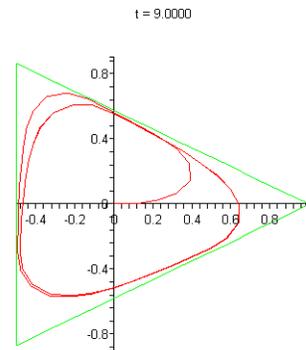
To derive the radius of this circle, assume that $0 < k \leq 1$ and parameterize the rabbit's path by $\mathbf{R}(t) = \langle \cos t, \sin t \rangle$. Then the fox's path can be parameterized by $\mathbf{F}(t) = k \langle \cos(t - b), \sin(t - b) \rangle$ where b is the angle that the fox lags behind. It is clear from the right triangle in the accompanying diagram that $b = \cos^{-1} k$ and $\lim_{t \rightarrow \infty} \|\mathbf{R} - \mathbf{F}\| = \sqrt{1 - k^2}$.



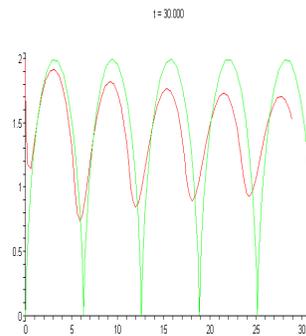
If $k < 0$, then the fox will run away from the rabbit. If the fox's initial position is the origin, then it will follow the negative y-axis for many values of negative k . If the fox's initial position is $(0,0.2)$, then it will exit in the second quadrant approaching a straight line.



The picture shown here is for a rabbit traveling along a triangle where $k = 0.5$. Although the rabbit instantly changes direction at the corners, the fox's path is a smooth triangular shape.



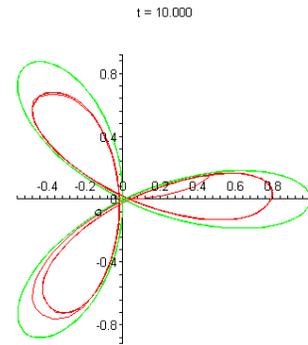
The picture here shows a rabbit following a sine wave where $k = 0.9$. If $0.6 < k < 1$, then the fox's path appears to follow a damped sine wave. If $k < 0.6$, then the fox lags further and further behind the rabbit and its path approaches the x-axis.



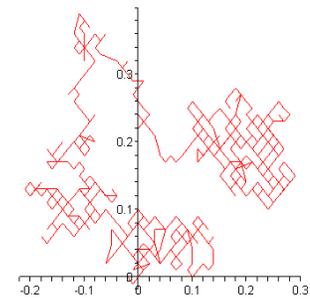
The path of a rabbit running on a cycloid

$\mathbf{R}(t) = \langle t - \sin t, 1 - \cos t \rangle$ is similar to the above case.

The picture shown here is for a rabbit running along a three-leaved rose $r = \cos(3\theta)$. The fox appears to be approaching a similar, but rotated slightly counter-clockwise curve.



I tried to have the rabbit randomly walk in the plane, but Maple's dsolve command gave an error when I tried to pass it the rabbit's path function.



Here are some ideas for further investigation:

- Find the equations for the attractors for sinusoidal and triangular paths for the rabbit
- Investigate this problem in three or higher dimensions.
- What if one fox is pursuing two rabbits?
- Incorporate delayed reaction time or anticipation into the model. If δ is the delay time, then the differential equation becomes $\mathbf{F}'(t) = k \|\mathbf{R}'(t - \delta)\| \frac{\mathbf{R}(t - \delta) - \mathbf{F}(t)}{\|\mathbf{R}(t - \delta) - \mathbf{F}(t)\|}$.
- Allow for speed of transmission. For example, if the fox uses sonar to detect the rabbit, then incorporate the speed of sound.
- Determine exit path dependence on fox's initial position for retreat curve when the rabbit is traveling in a circle.
- Find critical k where fox cannot keep up with a sine wave.

Sources

- “Differential Equation with Application and Historical Notes” by George F. Simmons (c)1972 by McGraw Hill
- <http://www.answers.com/topic/curve-of-pursuit>
- “In Pursuit” by Peter M. Gent, April 17, 1999
<http://online.redwoods.cc.ca.us/instruct/darnold/StaffDev/assignments/pursuit.pdf>
- “Pursuit Curves and Matlab” by Peter M. Gent
<http://online.redwoods.cc.ca.us/instruct/darnold/deproj/Sp98/PeterG/>
- “Pursuit Curve” <http://mathworld.wolfram.com/PursuitCurve.html>
- “The MacTutor History of Mathematics Archive” <http://www-groups.dcs.st-and.ac.uk/~history/>

Biography

Michael Lloyd received his B.S in Chemical Engineering in 1984 and accepted a position at Henderson State University in 1993 after earning his Ph.D. in Mathematics from Kansas State University. He has presented papers at meetings of the Academy of Economics and Finance, the American Mathematical Society, the Arkansas Conference on Teaching, the Mathematical Association of America, and the Southwest Arkansas Council of Teachers of Mathematics. He has also been an AP statistics consultant since 2001 and a member of the American Statistical Association.