VECTOR ALGEBRAIC THEORY OF ARITHMETIC: Part 2

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ABSTRACT: Virtually unrecognized by curricular educators is that the theory of Arabic/base-number arithmetic relies on equivalence classes of whole-scalar vectors. Sooner or later the recognition will catalyze a major re-formation of scholastic curricular mathematics – and also of the mathematics education of all teachers of mathematics. Part 1 of this survey appears in the 2005 Proceedings of the MAA's OK-AR Section Meeting. It sketches the vector-algebraic foundations of the Hindu-Arabic arithmetic for whole-number calculations – and that theory easily generalizes to include all base-number systems. As promised in Part 1, this Part 2 sequel outlines that theory's extension to cover the whole-scalar vector arithmetic of fractions – which includes the arithmetics of finite decimal-points and of all other n-al points. Of course, the extension to infinite-dimensional whole-scalar vectors leads off to the n-al arithmetics for the reals.

SUMMARY OF PART 1.

Part 1 opened with an alphabetic construction of the Peano line of simple Arabic digit-strings – devoid of any reference to numbers – and extended that construction to achieve an **Arabic numbers** kind of whole-numbers system. Part 1 concluded by disclosing how the whole-scalars, vector-algebraic structure which underlies the Arabic arithmetic for whole numbers can serve as a mathematical basis for greatly improving the instructional effectiveness of curricular education in the arithmetic of whole numbers.

That development described how the geometric-vectors – along the alphabetic line of simple Arabic digit-strings – yield whole-number additions/subtractions, and yield scalar multiplications/divisions by repeated additions/subtractions. (The geometric-vector definitions construct those operations simply by **counting spaces** along the scale of any Peano line.) Imposing those operations onto that line suffices for the resulting structure to satisfy the usual axiomatic definition for systems of whole numbers.

Thereby, the line of simple Arabic digit-strings becomes its own system of whole numbers (as well as being serviceable as a "numerals" vocabulary for any other whole-numbers line). Accordingly, once that concept of **Arabic numbers** has been achieved, even the very young can freely use such numbers as the scalars for finite dimensional vector spaces. Even before knowing how to perform any of the arithmetic calculations, children can rely on measuring tapes or hand-held calculators **to count the spaces**. They so own a line of whole-numbers line as needed for achieving the vector algebra on which Arabic arithmetic depends. The usual theorems for calculating with Arabic numerals then follow from the vector operations.

In Part 1, the most primitive whole-scalars vector spaces were called, "**inventory spaces**". Those are finite-dimensional, and become whole-scalar "**measurement spaces**" after they are duly partitioned by "**place-value equations**" among their unit vectors. In turn, examination of measurement spaces reveals that the familiar arithmetic of Arabic numerals requires even a far more complex whole-scalar vector-space, and relies on much higher developmental levels of mathematical sophistication within the learners. It means that early childhood, calculator-assisted, thorough learning about the vector-algebra of commonplace measurement systems should precede instruction in the far more sophisticated arithmetic of Arabic numerals.

While inventory and measurement spaces are finite-dimensional, the Arabic numerals are (singly-) infinite-dimensional vectors – normally truncated into finite **simple** forms. Generalizing from measurement-spaces, the Arabic numerals are seen to be but the **proper** vectors within equivalence classes of the **base-ten partition** of the singly-infinite dimensional, whole-scalars vector space. Within those base-ten cells the “carrying” and “borrowing” operations also use the **improper base-ten numerals** on which the Arabic arithmetic relies.

Thus Part 1 discloses that Arabic arithmetic is simply vector algebra, performed within the base-ten partition of the space of whole-scalar, singly infinite vectors – combined with intra-cell, vector-conversions to, and from their equivalent Arabic numerals. Beyond the vector operations, as such, is the operation of polynomial multiplication (with conversions) – whereby the arithmetic of the Arabic numerals is seen as taking place within **the base-ten partition of the space of whole-scalar polynomials**.

If the constructions in Part 1 have previously been published, elsewhere, such documents are not easy to find. Even if so, strictly academic interests in that much of the vector-algebra of arithmetic lie only within the foundations of mathematics. In that arena, the Part 1 constructions have their places among the classical (cardinal and ordinal) constructions of the wholes, the difference-pairs construction of the integers, the planar constructions of complexes, the quotient-pairs construction of the fractions, the planar constructions of the rationals, and the n-al constructions of the reals. The same is true of the additional constructions in Part 2. However, these two papers are presented far less for purposes of making original contributions to the foundations of mathematics than to shed new light on what mathematical knowledge is needed for teaching and especially for curriculum reform.

Each year, millions of humans are being educated in the arithmetics of whole numbers, decimal numbers, and more – often in troublesome ways that mathematically make no common sense – while the learners (and their teachers) are quite unaware that they actually are using a humanly natural version of vector algebra. Partitioned, whole-scalars vector spaces are not currently being used as mathematical foundations for prevailing school curricula, and are not yet seen even in the teacher-education textbooks in mathematics. Instead, the traditional mathematical basis for curricular instruction in arithmetic is mathematically shabby, weak, and confused – resulting in woefully ineffective learning and instruction, which undermines students’ mathematical growth throughout their years in school.

Moreover, schools now are being subjected to growing pressures – and also to strong skepticism – about pressing “grade 9” algebra into lower grades. So, this mathematical news that Arabic arithmetic actually is based on a mathematically rigorous, “kindergarten” version of vector algebra opens the way for a major algebraic re-formation of the early school curriculum through arithmetic. That pre-destined reformation surely will begin within a decade, if not sooner.

So arise two questions: one mathematical, and the other, instructological. First, does that vector-algebraic development of Arabic arithmetic extend, in an equally natural way, to include the usual arithmetics for the fractions? Yes – and that is what the rest of Part 2 is about.

Then, what does such a mathematical extension say, if anything, about a beneficial, vector-algebraic re-formation of the core-curriculum through fractions? The pragmatic need is to make the arithmetic of fractions more common-sensible to the learners. For sure, at least part of the vector theory can be advantageously invoked for improving school curricula. However, the vector-algebra of the whole numbers and fractions must become a compulsory topic in the education of all teachers of mathematics. Of course, there is always the danger that some over-zealous missionaries could worsen the scholastic situation by pressing the formal theory beyond the limits of the learner's own level of mathematical maturity.

SKETCH OF THE VECTOR ALGEBRAIC THEORY OF FRACTIONS

Classically, the term, "fractions" refers to all number-systems that meet an axiomatic definition for systems of fractions. It makes no sense to call them "non-negative rationals" without first having access to the concepts of "negative-ness" and of "rational-ness". The "new math" reformers of the 1960's unfortunately chose to use the term, "fractions" for all quotient-formulas that are expressed in the "over" format – regardless of the natures of the top and bottom objects. But the purpose of this paper is to provide mathematical clarification of the arithmetic for the number system – for which the label, "fractions" suffices.

In this vector-theory of the fractions, the major focus is on construction of the fractions within the setting of the infinite dimensional, whole-scalars vector space. Of course, this paper is intended for mathematicians as educators, so the basics of vector algebra are re-viewed only when doing so might provide some additional perspective on the algebraic re-formation of the core curriculum. For brevity, the construction is only sketched, and details are left to the readers.

The vector vocabulary for fractions is a generalization of the standard decimal-point vocabulary for the decimal numbers – and of every other n-al vocabulary for the n-al numbers. In each of those, their *proper* numerals are strings of placed whole-number "digits", with each string representing its own *conversion* equivalence class. Their respective arithmetics derive from using vector (and polynomial) operations across those classes, combined with various kinds of intra-class conversions. The vector theory of fractions is likewise developed from the space of doubly-infinite vectors having only whole-number coefficients, excluding use of sequences that are infinitely non-zero. [Of course, allowing the latter would lead to the continuum, which is not quite we want "fractions" to mean.]

The purpose of this extension is to shed new mathematical light onto the mathematical foundations for the core-curricular treatment of fractions. This presentation should not be construed as a recommendation as to whether or not novice students should learn the fractions as infinite-dimensional vectors. However, it presently appears that the vector theory of fractions is the optimal context for providing educators with fundamental mathematical knowledge needed for effectively teaching fractions.

The vector theory of fraction arithmetic is a **quantitative theory**. It views a fraction's ("simple") **monomial numerals** as being whole-scalar quantities, and views the remaining numerals as being **combinations of quantities**. That perspective conceptually contrasts with the (a-over-b) quotient-pairs. However, the vocabulary of numerical quotient-pairs is well entrenched in the culture and cannot be neglected in the curriculum. So, one role of the vector theory is to greatly increase the learners' conceptual understanding of traditional notions – and this paper sketches a mathematical backdrop for how to do so.

The key to the vector construction of fractions is that the **phonics** for the classical "**over**" symbols for fractions comes directly from the **quantities** used in inventory-spaces – whereby "3/4" is often pronounced as "3 fourths", meaning "3 of the 4ths". That (phonic) quantity could just as well be written as "3•f(4ths)" or "3(4ths)" or "3F". The "3(¯4¯)" expression, introduced by the UICSM curriculum designers, likewise speaks of quantities – and it can effectively assist the transition from "3(4ths)" to "3/4". [There is a notable conceptual difference between the meaning of "3(4ths)" and that of "3÷4". Equating those meanings requires a special theorem. Conceptually, "4ths" comes not directly from dividing 1 by 4, but from first factoring 1(Whole) into 4(4ths).]

As with any quantity, each fraction quantity consists of a **denomination** (that applies to all things of its kind), with a (coefficient) **numerator** depicting an amount of that kind – nD, as with "3F(rogs)". [Many years ago, the schools called such nD quantities, "denominate numbers".] The phonic vocabulary presents the fraction denominations essentially as **nths** – which presently are not included in the read/write language of the core curriculum. The vector theory focuses directly on the nths denominations, rather than relying on denomination-identifying *denominators* – the "demon-ators" whose visual displays so confuse students. The nths vernacular allows the harmless, but often helpful frivolity of occasionally speaking in terms of 1ths, 2ths, 3ths, and 5ths.

Empirically, nths are things, and each numerator speaks of whole number of things of that kind. As normally is done with the Arabic numerals and with the decimal-point numerals, the fraction denominations, too, can be expressed as sequential **places**. Thereby, each fraction quantity is expressed as a whole-number (the numerator) occurring in a particular (denomination) place. Thus, tuples of whole numbers can represent combinations of fraction quantities – once it is decided which which place represents which denomination. [That is how the n-al systems are distinguished from each other.]

The whole-numerals vector spaces and whole-measurements vector spaces (aired in Part 1) provide a conceptual springboard. Within the doubly-infinite, whole-scalars vector space are attended only those vectors whose coefficients are finitely non-zero. One of the denominations (i.e. the unit vectors) is designated as "wholes", W. In such whole-denominations spaces. each unit vector is a "whole" denomination whose **place-value** is some natural multiple of W. The units often are called by their respective multipliers, as with "5's" or "10's" or "2's", or '"20'S" etc . In that sense, each of those non-W denominations is "larger" than W, and the units are larger/smaller linearly ordered. In left-to-right languages, larger denominations are placed left of smaller denominations, so that each proper numeral or measurement converges from left to right. The **place-value equations** are set in that order – so that moving the unit's 1 by one place to the left or right equates with passage to the next higher or lower denomination. That conditions forces the W-unit to have its 1 in the right-most place. The place-value equations partition the vector space into cells of equivalent numeral/measurement vectors. Each cell contains a whole multiple of W, and those multipliers yield a Peano ordering of the partition's cells. The whole-numerals spaces differ from the whole-measurement spaces primarily in that the former are infinite dimensional, allowing for each cell to be "properly" represented by a constituent vector whose tuple of coefficients conform to bounds imposed on each place, by the place-value equations.

The vector construction of the fractions is similar. Again, the doubly-infinite, whole-scalars vector space provides the subspace of vectors whose coefficients are finitely non-zero. Also again, one denomination (i.e. unit vector) is designated as "wholes", W. But the place-value equations now are set so that for each non-W unit that is not a natural multiple of W, the W is a natural multiple of that unit. Units of the latter kind (including W) are that space's "fraction denominations". Again, larger/smaller denominations are defined with regard for their natural multipliers, and placed so that smaller denominations are right of larger ones. Moving a unit's 1 by one place equates with shifting to the next higher or lower denomination.

A **full** system of fractions uses all wholes and all fraction denominations. The place-value equations partition that vector space into cells of equivalent fraction vectors – each cell being a **fraction** whose constituents are infinite-dimensional **fraction-vectors**. [For simplicity while developing the fraction space, it suffices to neglect the non-W whole denominations. Thereby, each fraction-vector is singly infinite, finitely non-zero, with the W-denomination in the left-most place.]

Each fraction vector is a linear combination of fraction denominations, and the fraction vectors' whole-number coefficients are the numerators for the denominations expressed as sequential places. Each fraction vector is singly-infinite, but being only finitely non-zero, may be expressed as a tuple of whole-number numerators. [Such truncation is commonly used with the decimal-point numerals.]

Just as happens with whole measurements or whole numerals, and also with finite decimal-points and other n-al points. the denominations for successive places may be chosen at will (as a "basis"). The chosen place-value equations are arbitrary, but it is conventional for larger denominations to be left of smaller ones. On that basis, each finitely non-zero sequence of whole numbers expresses, left to right, a series that converges to its limiting sum.

Whatever fraction denominations are used, each fraction vector has W as its first place unit, (1, 0, 0 ....). Each of the other nths denomination comes from factoring W, by n, so that n(nths) = 1W. [Conceptually equating that factoring with division takes some doing.] Thus, in the full space, [2(0,1,0,0....)=1W] sets that unit vector as 2ths ... while in the decimal-point system, [10(0,1,0,0....)=1W] sets the second place unit as 10ths.

The arithmetic of fractions is about combining the (conversion cell) fractions through using vector operations and intra-cell conversions to achieve vectors that nicely identify the resulting fractions. The usual (and very natural) vector operations are class-consistent. But as usual with vector spaces, the vector operations do not automatically give a "multiplication" of vectors. Rather, the "multiplication" of fraction quantities comes from the [n(nths) = 1W] place-value equations that partition the vector space.

Closely related to such "multiplication" is the challenge of organizing such fractions (as conversion-class cells) into a linearly ordered system (if you wish, a "number line" of fractions). In the whole-measurement systems and whole-numeral systems, each vector "evaluates" as a whole multiple of the whole-unit, W=(...0,0,1), and the Peano ordering of their equivalence classes inherits from that of the whole numbers. Although the whole-fraction (cells) can likewise be aligned by using their w(1,0,0...) constituents, , the alignment of the remaining cells is not quite so simple. [In each family of n-al fractions, its reduced/proper vectors are alphabetically aligned, thereby aligning their respective cells. But not so for the full family of fraction cells.]

The key to the alignment is that the [n(nths) = 1] place-value equations also yield **quantity-conversion factors** – as with 4(4ths) = 12(12ths) yielding a **conversion factor** of 3. [Although the curriculum heavily relies on those conversion factors, it normally does not explicitly attend them. Rather, the visible denominators obscure the roles of the quantity-conversion factors.] From those conversion factors come the operations of **fractioning** fraction-quantities into equivalent ones, by using conversion factors as "cut numbers" ["cut each of the 3(4ths) in 2, to get into 6(8ths)"] ... and inversely of **fusing** quantities [fuse 6(8ths) by 2's, to get 3(4ths)]. But apart from such cut/fuse conceptual under-standings, **the condition that nths of mths gives (nm)ths** is a matter of mathematical definition, rather than of mathematical conclusion.

Within the full space, the quantity-conversion factors allow each fraction-vector to be *evaluated* as a single fraction-quantity – "simply" expressed as a fraction **quantity-vector** having at most one non-0 numerator. Each cell has its own infinite succession of denominations as used by its own quantity vectors. Among those quantity vectors, one has the largest denomination.

For each pair of cells, there is a sub-succession of denominations that are used **in-common** by each cell's quantity vectors. The numerators used for those in-common denominations suffice for comparing those two fraction (cells). Of course, that condition generalizes for the linear ordering of each finite family of fractions. A contrapositve argument excludes any exceptions for linear ordering of all such fractions.

Conversion Tools: the conceptual need is to **locate** each fraction (cell) as simply as possible, along the line of fractions, especially after executing some vector operations. For such purposes some kinds of fraction vectors suffice better than others. The intra-cell conversions of fraction vectors are used primarily for so locating the fractions – or for expediting calculations with the fractions.

A non-0 fraction's quantity-vectors are the **simple** vectors in that cell. A fraction vector is **simplified** when converted to a simple equivalent (that uses only one denomination). Whenever a collection of simple vectors share a single denomination, that denomination can be used for locating the simple vectors' fraction cells relative to each other.

Some kinds of **reduction** also facilitate location of fractions. Following the usual scheme with whole measurements, any reduction of a fraction vector is its conversion to an equivalent vector having a lower sum of the numerators [e.g. 19(28ths) = 0W+1(2th)+1(6th)+1(84th)]. The reduction is done by using remainder division on some numerators, to **carry** their quotients so as to **make** increments for numerators of higher denominations. [The conversion factor between 28ths and 2ths is 14; division of 19 by 14 yields 1 for carry onto the 2ths place, with a remainder of 5(28ths). That remainder is less than 1(3th), etc., down to 1(6th), etc. – yielding 1 for carrying into to the 6ths place.]

An **irreducible** vector admits to no such conversion [e.g.0W+ 1(2th)+1(6th)+1(84th) and 1(11th)]. Invariably, those are convergently increasing series. [Further below, some special cases are aired.]

Inverse to the reductions of the vectors are their **expansions** done by **borrowing** addends from some numerators, and using multiplications to **break** those, so as to increase the numerators for smaller denominations. [The familiar illustration is the case of "breaking" some U.S. (fraction) coins into smaller denominations.]

As a special case, **simple reduction** is the conversion of simple fraction vectors to equivalent simple vectors having smaller numerators. Each fraction (cell) has exactly one **simply irreducible** fraction vector – a simple vector that cannot be reduced to another simple vector. Reduced simple vectors are easier to locate than are non-reduced simple vectors.

Also useful for purposes of locating fractions are their **mixed** fraction vectors. [The word, "mixed" is used in deference to traditional curricular vocabulary.] Each is the sum of a whole-vector and one other simple vector. All simple vectors (including all whole vectors) are mixed, but the interesting cases have non-0 W-numerators and one other non-0 numerator. Each fraction conversion cell has a maximum numerator for its W-place. The cell vectors having that W-numerator are **proper** because they show where the fraction falls in reference to the whole fractions. All other mixed vectors have smaller W-numerators, suggesting that the fraction is in a whole-scale space below where it actually is. That is what makes them **improper**. So a simple vector is improper if its non-zero quantity is above or equal to 1W.

When the non-W denomination for mixed vectors is held constant, the equivalent mixed vectors constitute a succession from the equivalent simple vector, to the equivalent proper vector. To the novice, such conversion chains can be quite illuminating. A reduced mixed vector is necessarily proper, with an irreducible simple component. Each cell has exactly one such vector, called the **standard** vector for that fraction. The standard vectors are the ones that most readily locate each fraction along the whole-fractions scale.

Yet another kind of reduction uses a pre-selected family of denominations to locate the fraction. [E.g. when the chosen basis is [1ths,2ths, 3ths,4ths], 19(28ths) is above 0W+1(2th)+0(3ths)+0(4ths)] and below 0W+1(2th)+0(3ths)+1(4ths)]. A more commonplace illustration is where 53(64ths) is reduced (for convergent reading on binary-scaled rulers or tapes) as 0W+1(2th)+1(4th)+0(8th)+1(16th)+0(32th)+1(64th).

FRACTIONAL MEASUREMENT SPACES

The general vector theory of fractions (as cells of infinite-dimensional vectors) nicely provides all of the usual fraction arithmetic. While a viable context for the mathematical enlightenment of adults, the beginnings of fraction education of young children must build from their own experiences with familiar systems that have material meanings to them. For those, the more natural threshold consists of the **fractional measurement spaces** that are used in every household or community: kitchen tools, U.S. coins, clocks, scaled measuring tapes/sticks, angle measurements, etc

Most of the commonplace measuring systems use vector spaces. Judicious instructional choices of such measurement spaces can enable students to incrementally grow into the full fraction vector space. Already, curricular use of fractional measurement spaces are sometimes seen even in traditional entries into the arithmetic of fractions – even though educators have been unaware of the vector spaces, as such. However, the traditional norm has been to present each measurement system as an isolated, tangential use of fractions, rather than a component of an encompassing, unifying and under-standing theory. Attending measurements as vectors enables students to transport and integrate knowledge about each system, to knowledge about all others and about fractions, in general.

Many of our commonplace systems of measurements use fractional denominations, and not always in obvious ways. Very often, fractional or not depends on what one chooses to call "wholes". In U.S. coins, "a quarter" is 1(4th) ... if W means $. Likewise, in fluid measures, "a quart" means the fractional denomination, 1(4th), when W means "gallon". But when W means "cup"; a quart is the whole denomination, 4W , while an "ounce" is the fractional denomination 1(8th). So, many of the whole-measurements spaces attended in Part 1 become fractional measurement spaces when the W-unit shifted to the left (without changing the unit-conversion factors).

Each fractional measurement space is a **finite-dimensional**, whole-scalars, vector space partitioned by place-value equations for the respective unit vectors. The unit vector designated as W is not the right-most place (else the space would be of whole-measurements).

The place-value equations can be quite arbitrarily chosen, as long as all are of the kind, n(nths)=1W. Among all of the units, larger denominations are to the left of smaller denominations,. Although the vector operations necessarily are class consistent, intra-cell reductions might not always be possible, making the arithmetics a bit messy, depending on how the denominations are chosen.

The culture's commonplace languages for fractional measurements rarely disclose the (tuple) reliance on placed numerators. There are some familiar exceptions: $5.89 is spoken in **[$ | ¢]** as **[5 | 89]** As seen on many digital clocks, 3:43:17 presents a fraction of a day. On maps, 38°17'43" is a fraction of a circle. In curricular mathematics, the decimal-point numerals often are attended with regard for places and their place-values (wholes, 10ths, 100ths, ...) ... even though the numbers are pronounced as mixed vectors [e.g. 2.976 as "2 and 976(1000ths)"].

In the curricular setting, however, representation of measurements by use of successive **places**, **[| | | | | |]** can greatly enhance the cohesiveness of the evolving theory. (For novice students, the use of data boxes better emphasizes the use of places than do parentheses.) The spaces being finite dimensional, there always is a smallest denomination. In some measurement systems, the W-unit must be detected from names of whole denominations, as with the yards-long inch-stick (W=in.) **[yds | ft | in | 2ths | 4ths | 8ths]** In **strictly fractional** measurement spaces , the W-denomination always is on the left: e.g. **| $ | 2ths | 4ths | 10ths | nickels | 100ths**] .

In **evaluative** spaces, fractional measurements are conceptually simpler than in **non-evaluative** spaces. In the evaluative cases, the family of place-value equations establishes that all denominations are natural multiples of the smallest one – as with the vector space for combinations of U.S. coins. For, in such spaces, each vector has an "inner product" value as a natural multiple of the smallest denomination. All such spaces also admit to reduction and expansion by carrying and borrowing, and to conversions to mixed and standard vectors, and always to in-common denominations.

As with whole-number measurements, a **regular** fraction measurement system is one in which each non-smallest unit is a scalar multiple of the next smaller unit. That space being finite-dimensional, every regular space also is evaluative. When a regular system also is **consistent**, the conversion factors between adjacent denominations are all the same – as with the commonplace "yardstick", whose denominations make it actually an "inch stick". When that conversion factor is n, each measurement space is actually an initial interval of the space of n-al fractions – decimal, binary, octal, et al. So, studies of the arithmetics of consistent vector systems of fractional measurement, such as **| W | 2ths | 4ths | 8ths | 16ths | 32ths |** , paves the way for conceptual understanding of the arithmetic of decimal-point numerals.

[Herein, a "**decimal point**" ("finite" or "infinite") is regarded as being a doubly infinite succession, using places as unit-vector denominations, with one unit as W, and with each unit&45;vector being ten times any next lower one, and having only a finite number of non&45;0 numerator to the left of W. The "proper" decimal points have only digit numerators in each place; but the "improper" ones are needed for calculations. Likewise for "binary points", "octal points", etc. But for a **fractional point**, W is the only whole-number denomination and the numerators are finitely non&45;0. The fractional denominations may be arbitrarily chosen, but the **full fractional points** use all fractional denominations.]

The space for U.S. coins exemplify **irregular** evaluative spaces Most of the commonplace fractional measurement spaces are regular. But studies of the irregular cases shed much preliminary light on how the arithmetic of fractions actually work. For example, consider "standard" additions within the space: **| W | 2ths | 3ths | 6ths | 8ths | 24ths |**. Even a richer laboratory is the irregular **complete space** for 24ths that uses all natural factors of 24, **| W | 2ths | 3ths | 4ths | 6ths | 8ths | 12ths | 24ths |** . Also worth exploring are the complete finite spaces that use successive fraction denominations. The vector algebra enables most of the complicated aspects of fraction arithmetic to be resolved within the limited confines of finite-dimensional measurement spaces. [Of course, the vector operations, themselves, do not yield the "multiplications" of fractions.]

SIMPLE FRACTION SCALES

The 2-dimensional, [W | D] fractional-measurement spaces provide perhaps the most viable portal into the arithmetic of fractions. D is any non-W denomination for which, for some non-1 n, n(D)=1W ... as with the [$(ilvers) | Q(uarters)] space. Such [S | Q] use of initials puts the vector-algebra language within easy reach even of those too young to spell the words. Notice that in each place, the additions/subtractions of quantities, and their scalar multiplications/divisions are fully natural – as with 5D+4D and with 3•f7D. So, that "phonic" format for fractional quantities circumvents the confusions that are induced by the appearance of numerical denominators.

Each such 2-space consists of **binomial** mixed vectors – as with 7W+5D – among which the vector operations are naturally inherited from inventory spaces. Exploration of several such spaces, separately, easily discloses several important theorems which later generalize to the full vector space for the fractions. Each space has one place-value equation, and it clusters the binomials into equivalence classes. Reduction is done by carrying; expansion is done by borrowing. The proper vectors are the fully reduced ones. The simple vectors are the evaluations, 0W+vD, which often can be reduced to whole vectors, kW+0D. The vector operations are class-consistent, yielding "proper" operations by invoking carrying/borrowing. The fraction cells are aligned by their simple constituents in the smaller denomination, and also by alphabetic ordering of their proper constituents. Although scalar multiplication persists, there is no vector "multiplication" until additional denominations are appended.

One nicety of such spaces is that each gives rise to its own **simple fraction scale** which essentially overlays the 2-space onto the "primary school ruler" scale of wholes. That is the one (sorely neglected) empirical model that nicely abstracts into the full line of fractions. As below, initial examination of each of several separate fraction scales naturally leads toward **joining** two or more scales, as illustrated by familiar measuring tapes/sticks. The line of fractions evolves as **the join** of all simple fraction scales.

The set of all whole multiples of a unit vector is a **dimension** or **axis** of the vector space. Each is a Peano line for that denomination – a discretely ordered line, with a min and no max. Along each such axis, adjacent points are separated by 2-point intervals – the (line-) **spaces** of that line. The alternating succession of quantities and spaces is a **Peano scale** in that denomination – whereon the multiples of the unit vector are the **marks** of the scale. So, the family of all such axes is a singly-infinite dimensional **Arabian stone-counter** – portions of which are viable empirical laboratories for representing the vectors.

As a fractional counter, the W-scale is on the left, and the place-value equations are of the form n(nths)=1W. The place-value equations establish that each mark on the W-scale equates with a (W') mark on each nths-scale. On the W-scale, successive marks are separated by a single space. But on each nths-scale, successive W'-marks are separated by n spaces, and by n-1 marks. In effect, each non-W nths-scale **fractures** each W-line-space into n line-spaces. [Notable, here, is the absence of any dependence on some kind of geometric "congruence" among a fraction scale 's own line-spaces. Rather, each of the intervals along each axis later acquire "lengths" as scalar differences of their terminals. The cross-dimension "congruence" comes still later, when the binary scales are joined by their in-common refinements.]

For a scientific laboratory, the "primary school" W-scale is extended as a family of simple&45;scale marked slats (e.g. from Venetian blinds) – with W-spaces chosen according to whatever works for what learners (often 1 decimeter whole-spaces). Passage to each 2-space is done by inserting additional nths-marks, in the mode of commonplace yardsticks. In effect, the nths scale thus is **overlaid** onto the W-scale. A wealth of theorems surface from studying just the simple fraction scales.

Each axis is **naturally refined** by using a natural **cut number** greater than 1, to shift to an axis of smaller denomination. The simple (binary) fraction scales reveal that each nths-dimension is a **refinement** of the W-dimension. That simple observation explains how fractions are natural "divisions" of whole numbers. The **scalar division** of any whole quantity is done by first **factoring** that quantity by using the divisor as a cut number, and then identifying the quota-number accrued by each of the parts. To divide 7W by 3, cut the W-scale by 3, to get the [W |3ths] scale – on which the W-scale's 7-mark equates with the 21-mark on the 3ths scale. The revelation is that the scalar division actually is done on the 3ths scale, as 7W÷ 3 =21(3ths)÷ 3 = 7(3ths). The condition that 7(3ths) = 7W÷ 3 might serve as a justification for using the "7÷ 3" formula to mean 7(3ths), but gaining conceptual understanding of that equation requires examining something like the simple fraction scale.

The conversion of 7W to 21(3ths) is a simple case of passing to an in-common denomination, but applying a cut number also causes a refinement of any simple scale to another (2-dimensional) simple scale whose place-value multiplier is a non-1 natural multiple of the earlier one, as with [W |3ths] → [W | 12ths]. The effect is essentially to **fraction** each of the spaces in the earlier one. Inversely, natural numbers can be used as a **fusion numbers** for passing to larger-denomination simple scales.

Along the way surfaces the fact that when the [W |nths] scale is cut by the number, m, the resulting refinement is the [W | (mn)ths] scale. So arises the "•f" cross combinations of the unit vectors, as **mths •f nths = (mn)ths** ... and a reason for calling such refinements the "multiplication" of denominations. Like the conversion of 7W to 21(3ths), each refinement is a simple case of passing to an in-common denomination. The more general case uses two or more simple scales.

Each pair of simple fraction scales has an infinite family of **in-common refinement** simple scales, one of which has the largest non-W denomination. With an adequate laboratory of simple scales (say, 2ths through 12ths), most of the basic-literacy theorems about using in-common denominations can be surfaced through study merely of simple fraction scales.

Collectively, the (binomial) simple fraction scales provide a strong conceptual standpoint and a global perspective on how the cut/fuse numbers work for converting among equivalent fraction vectors. The cut/fuse numbers already are widely used in the core-curriculum, but – much to the conceptual loss of the students – are not explicitly attended, much less being interpreted on fraction scales.

Evidently, premature curricular injection of numerical denominators (in the standard "over"-vocabulary for fractions) has long hidden the underlying, more conceptual vector algebra of fractions. The visual display of numerical denominators obscures the equivalence classes of vectors – and conceals the fact that "doing fractions" is an arena of doing algebra. The phonics of the simple fraction scales call for conceptually expressing fractions as vectors, in quantity terms of nths, rather than relying on "demon-ators". Well before focusing on numerical denominators – much headway can be made into the arithmetic of fractions, by exploring the vector algebra of simple scales. Indeed, considering the nationwide dilemma with widespread non-comprehension of fractions, it is amazing that the core curriculum has so badly neglected student literacy with "rulers".

AD INFINITUM

The family of simple fraction scales gives rise to various kinds of not-so-simple scales whose exploration leads to achievement of the full line of fractions. The familiar **| W | 10ths | 100ths | 1000ths | ... |** and **| W | 2ths | 4ths | 8ths | 16ths | ... |** and n-al spaces present successions of axes whose scales are got by *consistently* using a single cut number to refine the last scale, to get still another scale to add to the space. Varying the cut number still *regularly* yields successive refinements, as with **| W | 12ths | 60ths | 360ths | ... |**

Each such refinement presents a **serial fraction scale** whereon refinements are successively overlaid onto their precursors.

In the simple-scale refinements of simple scales, the cut numbers are used for fractioning the spaces on fraction scales. That naturally leads to extending the 2-space to a regular 3-space, and the passage from simple scales to **serial fraction scales**. As is seen on commonplace yardsticks, a **regular series** of scales is achieved by successive overlaying, onto a fraction scale, a refinement of its smaller denomination. For the novice, the progressive densification of the series is more readily seen when simple scales are blocked together as with parallel-ruled paper. Although yardsticks and meter sticks present even **consistent series**, construction of **inconsistent regular series** of fraction scales better sets the stage for the **irregular series** on which the arithmetic of fractions depends. Of course, the existence of infinite series of that kind guarantees the existence of dense lines of fractions.

However, not all **complex fraction scales** are that well behaved. When the [W |3ths] scale is overlaid onto the [W |2ths] scale, the resulting complex scale is the [W |2ths |3ths]. But the [W |2ths |3ths] complex is a bit of a mess.

All complex fractional scales are finite-dimensional, got by overlaying three or more axial scales onto each other (one being the W-scale). Always, the composite is a a Peano scale whose marks are separated by one or more line-spaces. Exploring the overlaid fraction scales can surface the need for **joining fraction scales** by use of in-common denominations.

The join of the [W |2ths] and [W |3ths] simple scales is the complex scale, [W |2ths |3ths |6ths]. While both the [W |2ths |6ths] and [W |3ths |6ths] scales are serial complex scales, the [W |2ths |3ths |6ths] join is not a series. Consider an irregular 3-dimension fraction measurement space, [W |nths | mths] Merely overlaying the [W |mths] scale onto the [W | nths] scale usually fails to provide a refinement of either. But the space can be extended into a [W |nths | mths| kths] space whose kths scale is an in-common refinement of both the nths scale and the mths scale. Since all marks from each of the first two are replicated on the third, the scalar operations along each of the first two can be combined as scalar operations along the third.

The lattice&45;theoretic logic of in-common denominations so surfaces through the simple-scales laboratory. So does the need for some way to use the n and m in order to find an adequate k ... and to get each of the two cut numbers ... and even to find a smallest such k (i.e. the largest such denomination). The natural beginning is to construct a table of the natural refinements of each, in search of an in-common one. [Conceptual understanding of the logic of using prime factorizations requires much higher sophistication.]

The space of all singly infinite, whole-scalar, finitely non-0 vectors has an infinitude of axes, each of which consists of all whole multiples of its unit vector, and is its own Peano line, and so provides its own Peano scale. Each finite selection of units/axes provides a finite-dimensional subspace useful for constructing a fraction-measurements space – when place-value equations establish the first unit as a W, and as natural multiples of each of the other units, respectively. But only the evaluative subspaces are certain to have scales that fully admit to the vector operations (and none fully to fraction multiplications).

While all of the above can be achieved through the vector-based, finite-dimensional measurement spaces, no one space of that kind can accommodate all of the fraction multiplications from within that space, much less all of the fraction-vectors. Finite-dimensionality eventually must be outgrown, even without losing finitely non-zero-ness. Even in the infinite-dimensional space of all “finite” fraction&45;points, operations on any finite set of them can be done in a finite-dimensional subspace, the extreme being the space that uses all fraction denominations down to the last one needed. Progressive passage to in-common refinements soon discloses that the full line of fraction-vector cells is dense.

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