Existence and Uniqueness Proof for nth Order Linear Differential Equations with Constant Coefficients

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Abstract

Standard textbooks in introductory differential equations (see [1], [2], [3], [4], [5]) expose the subject by emphasizing more on the techniques and applications and cutting out most of the theoretical exposure. As a result, many of the theoretical results are presented without proofs. A collection of class lecture notes have been written to examine whether including the proofs will be suitable for the beginning sophomore students taking a course in introductory differential equations. (See [8] and [9]) Surprisingly, we found out that the proofs provide a wealth of information very useful for students in mathematics and the applied sciences. In this paper, we take an excerpt from these notes. We investigate the conditions on the coefficients of an nth order linear differential equation for which a reasonable proof of existence and uniqueness of solutions can be established.

1 Introduction

When teaching an introductory course in differential equations one notices that most theorems of the theory are almost always stated without proofs in elementary differential equations textbooks following calculus. In this paper, we discuss the importance of including the theoretical justifications of the majority of the theorems. A collection of class lecture notes have been written

to examine whether including the proofs will be suitable for the beginning sophomore students taking a course in introductory differential equations. (See [8] and [9]). For references, we used the books [1], [2], [3], [4], [5], and [6]. The first five are relatively suitable to students at the sophomore level who had a calculus course. They are "suitable" because they avoid going through the justification of most of the theorems that they discuss and they stress more on the drilling aspect of differential equations rather than the reasoning and the drilling that follows from it. The last fits the bill but is more designed for advanced students.
In writing the notes mentioned earlier, we found out, to our surprise, that most of the proofs are relatively constructive and understandable, and provide a wealth of information that can be useful for both instructors and students:
(1) From the instructor side, knowing the theory behind the results will enhance the teaching of the topics.
(2) From the student side, knowing the theory prepare them to be more research oriented individuals and thus more creative and with strong reasoning skills compared to individulas whose only interest is just to apply a collection of techniques.
To emphasize the importance of theoretical discussion, we consider the question of existence and uniquenss of solutions to linear differential equations. Typically, one starts by discussing this question for first order linear differential equation. A method of proof is the integrating factor method which we describe next.

2 The Method of Integarting Factor for First Order Linear Differential Equations

One way to establish the existence of a unique solution to the initial value problem

$$y' + p(t)y = g(t), \quad y(t_0) = y_0 \tag{1}$$

where $p(t)$ and $g(t)$ are continuous in $a < t < b$ is by applying the method of integrating factor.
For the existence of a solution we proceed as follows: Since $p(t)$ is continuous

then by the Second Fundamental Theorem of Calculus the function

$$\int_{t_0}^{t} p(s)ds$$

is differentiable with derivative

$$\frac{d}{dt}\int_{t_0}^{t} p(s)ds = p(t), a < t < b$$

Let

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds}$$

From this, one can notice that Equation (8) can be written as

$$(\mu(t)y)' = \mu(t)g(t)$$

Integrating this last equation to obtain

$$\mu(s)y(s)|_{t_0}^{t} = \int_{t_0}^{t} \mu(s)g(s)ds$$

Thus,

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(s)g(s)ds$$

or

$$\mu(t)y(t) - y_0 = \int_{t_0}^{t} \mu(s)g(s)ds$$

Divide the last equation by $\mu(t)$ to obtain

$$y(t) = \frac{1}{e^{\int_{t_0}^{t} p(s)ds}}\int_{t_0}^{t} \mu(s)g(s)ds + \frac{y_0}{e^{\int_{t_0}^{t} p(s)ds}} \tag{2}$$

As for the uniqueness, we suppose that $y_1(t)$ and $y_2(t)$ are two solutions of (8). Let $w(t) = y_1(t) - y_2(t)$ for any $a < t < b$. We will show that $w(t) \equiv 0$ for all $a < t < b$. First, we show that $w(t)$ satisfies the homogeneous equation

$$w' + p(t)w = 0 \tag{3}$$

Indeed,

$$w' + p(t)w = (y_1' + p(t)y_1) - (y_2' + p(t)y_2) = g(t) - g(t) = 0.$$

Multiply Equation (3) by $e^{\int_{t_0}^t p(s)ds}$ to obtain

$$\left(e^{\int_{t_0}^t p(s)ds} w\right)' = 0$$

Now integrate both sides and then solve for $w(t)$ to obtain

$$w(t) = Ce^{-\int_{t_0}^t p(s)ds} \tag{4}$$

But $w(t_0) = y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0$ so that $C = 0$. Hence, $w(t) \equiv 0$ for all $a < t < b$ or $y_1(t) = y_2(t)$ for all $a < t < b$.

Next, comes the question of existence and uniqueness for nth order linear differential equation with initial conditions. Here, most books avoid providing a proof because it's beyond the scope of a course in elementary differential equations. So, it makes sense to think about whether the method of integrating factor discussed for first order linear differential equation can be extended to higher order equations, a topic that we discuss next.

3 Extended Integrating Factor Method

We consider the following nth order linear differential equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t) \tag{5}$$

with initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y'_0,\ \ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)},\ a < t_0 < b \tag{6}$$

where the functions $p_0(t),\ p_1(t), \cdots,\ p_{n-1}(t)$, and $g(t)$ are continuous in an open interval $a < t < b$.
The above initial value problem can be transformed into a first order system. This is done by introducing the variables

$$x_1 = y,\ x_2 = y',\ \ \cdots,\ x_n = y^{(n-1)}.$$

In this case, we have

$$\begin{array}{rcl} x'_1 & = & x_2 \\ x'_2 & = & x_3 \\ \vdots & = & \vdots \\ x'_{n-1} & = & x_n \\ x'_n & = & -p_{n-1}(t)x_n - \cdots - p_1(t)x_2 - p_0(t)x_1 + g(t) \end{array}$$

Thus, we can write the problem as a system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}' + \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \\ p_0 & p_1 & p_2 & p_3 & \cdots & p_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

or in compact form

$$\mathbf{x}'(t) + A(t)\mathbf{x}(t) = \mathbf{b}(t), \quad \mathbf{x}(t_0) = \mathbf{y}_0 \tag{7}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \\ p_0 & p_1 & p_2 & p_3 & \cdots & p_{n-1} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{b}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}, \ \mathbf{y}_0 = \begin{bmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}$$

Note that if $y(t)$ is a solution of (5) then the vector-valued function

$$\mathbf{x}(t) = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

is a solution to (7). Conversely, if the vector

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

is a solution of (7) then $x_1' = x_2,\ x_1'' = x_3, \cdots, x_1^{(n-1)} = x_n$. Hence, $x_1^{(n)} = x_n' = -p_{n-1}(t)x_n - p_{n-2}(t)x_{n-1} - \cdots - p_0(t)x_1 + g(t)$ or

$$x_1^{(n)} + p_{n-1}(t)x_1^{(n-1)} + p_{n-2}(t)x_1^{(n-2)} + \cdots + p_0(t)x_1 = g(t)$$

which means that x_1 is a solution to (5).
To carry out the method of integrating factor successfully to equation (7) requires that we have

$$\left(e^{\int \mathbf{A}(t)dt}\mathbf{x}(t)\right)' = e^{\int \mathbf{A}(t)dt}\mathbf{x}'(t) + \mathbf{A}(t)e^{\int \mathbf{A}(t)dt}\mathbf{x}(t) \tag{8}$$

where for any square matrix $\mathbf{A}(t) = (a_{ij}(t))$ we define

$$\int_{t_0}^{t} \mathbf{A}(s)ds = \left(\int_{t_0}^{t} a_{ij}(s)ds\right).$$

But Equation (8) is valid only if

$$\frac{d}{dt}\left(e^{\int_{t_0}^{t} \mathbf{A}(s)ds}\right) = \mathbf{A}(t)e^{\int_{t_0}^{t} \mathbf{A}(s)ds}$$

and this last equation is valid when

$$\mathbf{A}(t) \cdot \int_{t_0}^{t} \mathbf{A}(s)ds = \int_{t_0}^{t} \mathbf{A}(s)ds \cdot \mathbf{A}(t)$$

and this equation leads to the following system of integral equations

$$\begin{array}{ccc} \int_{t_0}^{t} p_0(s)ds & = & p_0(t)(t-t_0) \\ \int_{t_0}^{t} p_1(s)ds & = & p_1(t)(t-t_0) \\ \vdots & & \vdots \\ \int_{t_0}^{t} p_{n-1}(s)ds & = & p_{n-1}(t)(t-t_0) \end{array}$$

A solution to the above system is when the functions $p_0(t),\ p_1(t),\ \cdots,\ p_{n-1}(t)$ are constant functions. Thus, one can prove the existence and uniqueness of solutions to nth order linear differential equations with constant coefficients using the method of integrating factor. The closed form of the solution is

$$\mathbf{x}(t) = e^{-(t-t_0)\mathbf{A}}\mathbf{x}(0) + e^{(t-t_0)\mathbf{A}} \int_{t_0}^{t} e^{-(s-t_0)\mathbf{A}}\mathbf{b}(s)ds.$$

4 Illustration

We next apply the method of integrating factor described in the above theorem to solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \; y'(0) = 0$$

In this problem, $n = 2$ so that $p_1(t) = 0$ and $p_0(t) = -1$ so that

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Hence,

$$-\int_0^t \mathbf{A}dt = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}$$

Now, one can easily see that for any nonnegative odd integer n we have

$$\left[-\int_0^t \mathbf{A}dt\right]^n = \begin{bmatrix} 0 & t^n \\ t^n & 0 \end{bmatrix}$$

and for nonnegative even integer n

$$\left[-\int_0^t \mathbf{A}dt\right]^n = \begin{bmatrix} t^n & 0 \\ 0 & t^n \end{bmatrix}$$

Thus,

$$e^{-\int_0^t \mathbf{A}dt} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} & \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \end{bmatrix} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

Thus,

$$\mathbf{x}(t) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From this we obtain the unique solution $y(t) = \cosh t$.

We conclude from the above discussion that a reasonable proof can be given for the existence and uniqueness of solutions to nth order linear differential equations. The proof extends the integrating factor method to a first order linear system representing the scalar differential equation. Does this extension works for any first oder linear systems? We tackle this problem next.

5 Integrating Factor Method for First Order Linear Systems

For simplicity of the discussion we limit ourselves to the case $n = 2$, that is we consider the initial value problem

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2 \end{aligned}$$

with initial conditions

$$y_1(t_0) = y_1^0, \quad y_2(t_0) = y_2^0.$$

In matrix form we have

$$\mathbf{Y}'(t) = \mathbf{A}(t)\mathbf{Y}, \quad \mathbf{Y}(t_0) = \mathbf{Y}_0$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

For the integrating factor method to work we need

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} \int_{t_0}^t a_{11}ds & \int_{t_0}^t a_{12}ds \\ \int_{t_0}^t a_{21}ds & \int_{t_0}^t a_{22}ds \end{bmatrix} = \begin{bmatrix} \int_{t_0}^t a_{11}ds & \int_{t_0}^t a_{12}ds \\ \int_{t_0}^t a_{21}ds & \int_{t_0}^t a_{22}ds \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

This leads to the system

$$\begin{aligned} a_{11}\int_{t_0}^t a_{11}ds + a_{21}\int_{t_0}^t a_{12}ds &= a_{11}\int_{t_0}^t a_{11}ds + a_{12}\int_{t_0}^t a_{21}ds \\ a_{12}\int_{t_0}^t a_{11}ds + a_{22}\int_{t_0}^t a_{12}ds &= a_{11}\int_{t_0}^t a_{12}ds + a_{12}\int_{t_0}^t a_{22}ds \\ a_{11}\int_{t_0}^t a_{21}ds + a_{21}\int_{t_0}^t a_{22}ds &= a_{21}\int_{t_0}^t a_{11}ds + a_{22}\int_{t_0}^t a_{21}ds \\ a_{12}\int_{t_0}^t a_{21}ds + a_{22}\int_{t_0}^t a_{22}ds &= a_{21}\int_{t_0}^t a_{12}ds + a_{22}\int_{t_0}^t a_{22}ds \end{aligned}$$

The integrating factor is successful here provided that either $a_{11}, a_{12}, a_{21}, a_{22}$ are constants or $a_{11} = a_{22}$ and $a_{12} = a_{21}$.

6 Concluding Remarks

As a conclusion of this presentation, we believe that a more recommended textbook in differential equations is the one that covers both the detailed-theoretical and drilling aspects of the subject at the sophomore level. The theoretical aspect should be included and should be left to the discretion of the reader to cover it or not.

" *I prefer a theorizer for my children education rather than a problem solver.*"
-C.G. Rota

References

[1] W. Kohler and L. Johnson, *Elementary Differential Equations,* Second Edition, Addison-Wesley, (2006).

[2] R. Nagle, E. Saff, and A. Snider, *Fundamentals of Differential Equations,* Addison Wesley, Sixth Edition, (2004).

[3] G. Simmons and S. Krants, *Differential Equations: Theory, Technique, and Practice,* McGraw-Hill, (2007).

[4] D. Zill, *First Course in Differential Equations,* Fifth Edition, Thomson Learning, (2000).

[5] W.E. Boyce and W.C. DiPirma, *Elementary Differential Equations,* Sixth Edition, Wiley Pub, (1997).

[6] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations,* New York, McGraw Hill (1955).

[7] W. Kaplan, *Advanced Calculus,* Third Edition, Addison-Wesley (1984).

[8] Marcel B Finan, *A First Course in Elementary Differential Equations,* Arkansas Tech University (2005).

[9] Marcel B Finan, *A Second Course in Elementary Differential Equations,* Arkansas Tech University (In Progress).

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