1. Divisible by 19.
Suppose that \(2x + 6y = 19m\). Then \(m\) is even; say \(m = 2k\). Then \(x + 3y = 19k\), and \(5x + 15y = 5 \cdot 19k\); so \(5x - 4y = 5 \cdot 19k - 19y = 19(5k - y)\). Conversely, suppose that \(5x - 4y = 19n\). Then \(5x + 15y = 19n + 19y = 19(n + y)\), and thus \(n + y\) is a multiple of 5; say \(n + y = 5r\). Then \(5x + 15y = 19 \cdot 5r\), and \(x + 3y = 19r\).

2. Intersections of quadratics.
The two graphs have infinitely many points in common. Since (1) factors as
\[
(2x + y - 4)(x - y) = 0
\]
and (2) factors as
\[
(2x + y - 4)(3x - y - 2) = 0,
\]
the graph of each is a pair of intersecting lines, and the two have the line \(2x + y - 4 = 0\) in common. (The correct statement is that the graphs of two quadratics have at most four points of intersection or they have infinitely many.)

3. No solutions in integers.
Suppose that \(m(m + 1) = n(n + 2)\). Upon adding 1 to each side we have
\[
m^2 + m + 1 = (n + 1)^2.
\]
But
\[
m^2 < m^2 + m + 1 < (m + 1)^2;
\]
i.e.,
\[
m^2 < (n + 1)^2 < (m + 1)^2.
\]
Thus if \(m\) is an integer, \(n\) is not.

4. Square populations.
We show that the 1980 population had to be \(5^2 = 25\) or \(499^2 = 249001\). Let \(n^2\) be the 1980 population. Then \(n^2 + 1000 = m^2 + 1\) and \(n^2 + 2000 = r^2\) for some integers \(m\) and \(r\). From \(m^2 - n^2 = 999\) we have
\[
(m - n)(m + n) = 3^3 \cdot 37.
\]
Since \(m - n < m + n\), we conclude that \(m - n = 1, 3, 9\) or 27. The corresponding values of \(m + n\) are 999, 333, 111 and 37, leading to the pairs
\[
(m, n) = (500, 499), \ (168, 165), \ (60, 51), \ (32, 5),
\]
respectively. Of these, the only ones for which \(n^2 + 2000\) is a square are \((500,499)\) and \((32,5)\). Thus \(n\) must be 499 or 5, and the 1980 population had to be \(499^2\) or \(5^2\).
5. An integer valued function.

It suffices to show it for nonnegative integers, because $f(-x) = -f(x)$. Here is a proof by induction. $f(0) = 0$ is an integer. Suppose that $f(n)$ is an integer. Then

$$f(n + 1) = \frac{1}{5}(n + 1)^5 + \frac{1}{3}(n + 1)^3 + \frac{7}{15}(n + 1)$$

$$= \frac{1}{5}(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) + \frac{1}{3}(n^3 + 3n^2 + 3n + 1) + \frac{7}{15}(n + 1)$$

$$= \left(\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n\right) + (n^4 + 2n^3 + 2n^2 + n) + (n^2 + n) + \left(\frac{1}{5} + \frac{1}{3} + \frac{7}{15}\right),$$

which is again an integer. By induction, $f(n)$ is an integer for all integers $n \geq 0$.


We may write

$$1 + \frac{1}{k^2 + 2k} = \frac{(k + 1)^2}{k(k + 2)},$$

so

$$\ln\left(1 + \frac{1}{k^2 + 2k}\right) = \ln\left(\frac{(k + 1)^2}{k(k + 2)}\right) = [\ln(k + 1) - \ln k] - [\ln(k + 2) - \ln(k + 1)],$$

and the series telescopes:

$$\sum_{k=1}^{n} \ln\left(2 + \frac{1}{k^2 + 2k}\right) = \sum_{k=1}^{n} \left([\ln(k + 1) - \ln k] - [\ln(k + 2) - \ln(k + 1)]\right)$$

$$= [\ln 2 - \ln 1] - [\ln(n + 2) - \ln(n + 1)]$$

$$= \ln 2 - \ln\left(\frac{n + 2}{n + 1}\right),$$

which converges to $\ln 2$ as $n \to \infty$.

7. Multiplying two arithmetic sequences.

The seventh term is $280$. Write the first three terms of the arithmetic sequences as $a - d$, $a$, $a + d$ and $b - e$, $b$, $b + e$. Then $1216 = (a - d)(b - e) = ab - bd - ae + de$, $1360 = ab$, and $1384 = (a + d)(b + e) = ab + bd + ae + de$. Adding the first and third of these equations and dividing by 2 gives $ab + de = 1300$, whence $de = -60$, and $bd + ae = 84$. The seventh terms of the arithmetic sequences are $a + 5d$ and $b + 5e$, so the seventh term of our product sequence is

$$(a + 5d)(b + 5e) = ab + 5(bd + ae) + 25de = 1360 + (5)(84) + 25(-60) = 280.$$
8. Probability that it is divisible by 11.

The probability is \(\boxed{11/126}\). There are 9! integers in which each of the nine digits occurs exactly once. We need to count those divisible by 11. Write the number

\[ n = a_810^8 + a_710^7 + a_610^6 + a_510^5 + a_410^4 + a_310^3 + a_210^2 + a_110 + a_0. \]

Modulo 11,

\[ n \equiv a_8(-1)^8 + a_7(-1)^7 + a_6(-1)^6 + a_5(-1)^5 + a_4(-1)^4 + a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 \]

\[ = (-1)(a_7 + a_5 + a_3 + a_1) + (a_8 + a_6 + a_4 + a_2 + a_0), \]

so \( n \) is divisible by 11 if and only if \( A \equiv B \pmod{11} \), where \( A = a_1 + a_3 + a_5 + a_7 \) and \( B = a_0 + a_2 + a_4 + a_6 + a_8 \). Then we have \( A + B = 45 \) and we need \( A - B = 11k \) for some integer \( k \). These two conditions imply \( 2A = 45 + 11k \) and \( 2B = 45 - 11k \). We see that \( k \) must be odd and \( |k| < 4 \). The possible values for \( 2A \) then are 12, 34, 56, 78. It is easy to see that neither 6 nor 78 can occur, so \( A \) is 17 or 28. There are 9 possible sets \( \{a_1, a_3, a_5, a_7\} \) with sum equal to 17, and 2 with sum 28. They are \( \{1, 2, 5, 9\}, \{1, 2, 6, 8\}, \{1, 3, 4, 9\}, \{1, 3, 5, 8\}, \{1, 3, 6, 7\}, \{1, 4, 5, 7\}, \{2, 3, 4, 8\}, \{2, 3, 5, 7\}, \{2, 4, 5, 6\}, \{4, 7, 8, 9\}, \{5, 6, 8, 9\} \). Thus, there are 11 choices for the set of 4 digits in \( A \), and the other 5 digits go into \( B \). This gives us \( 11(4!)(5!) \) permutations of the 9 digits yielding a multiple of 11, and the desired probability is

\[
\frac{11(4!)(5!)}{9!} = \frac{11 \cdot 4 \cdot 3 \cdot 2}{6 \cdot 7 \cdot 8 \cdot 9} = \frac{11}{126}.
\]

9. Length of a segment.

We show that \( AC = \sqrt{2} \). Let \( x = AC \), \( y = BC \), and \( z = BD \). By the law of sines,

\[
\frac{z}{\sin BCD} = \frac{1}{\sin 30^\circ} = 2
\]

and

\[
x = \frac{x}{\sin 90^\circ} = \frac{1}{\sin(180^\circ - BCD)} = \frac{1}{\sin BCD} = \frac{2}{z}.
\]

By the law of cosines applied to triangle \( ABD \) we have \((1 + x)^2 = 1^2 + z^2 - 2z \cos 120^\circ = 1 + z^2 + z\). Substitution of \( x = 2/z \) gives \((1 + 2/z)^2 = 1 + z + z^2\); i.e., \((z + 1)(z^3 - 4) = 0\). The only positive real root of this equation is \( z = \sqrt[4]{4} \), and hence \( x = 2/z = \sqrt{2} \).
SECOND SOLUTION

Using the law of cosines applied to angle $BCD$ in triangle $BCD$ we obtain

$$z^2 = 1 + y^2 + 2y^2/x. \quad (1)$$

Apply the law of cosines to angle $B$ in the triangle $ABD$ to get

$$(x + 1)^2 = 1 + z^2 - 2 \cdot 1 \cdot z \cos 120^\circ = 1 + z^2 + z,$$

and thus

$$x^2 + 2x = z^2 + z. \quad (2)$$

Substitute $y^2 = x^2 - 1$ (Pythagorean Theorem) into (1) to get

$$z^2 = x^2 + 2(x^2 - 1)/x,$$

so

$$z^2 = x^2 + 2x - 2/x. \quad (3)$$

From (2) and (3) it follows that $z = 2/x$. Substitute $z = 2/x$ into (3), giving

$$\left(\frac{2}{x}\right)^2 = x^2 + 2x - 2/x.$$ 

Now multiply both members by $x^2$ to get $4 = x^4 + 2x^3 - 2x$; i.e., $(x^3 - 2)(x + 2) = 0$. As $x$ is positive it follows that $x = \sqrt[3]{2}$.

10. 2019 sums of consecutive integers.

The smallest is $N = 3^{100} \cdot 5^4 \cdot 7 \cdot 11$. We note that

$$N = a + (a + 1) + \cdots + (a + n) = (n + 1)(2a + n)/2 \iff 2N = (n + 1)(2a + n).$$

Moreover, exactly one of $n + 1$ and $2a + n$ is odd, and each factor is greater than 1. Conversely, every odd divisor $d$ of $2N$ greater than 1 uniquely determines values of $n$ and $a$ so that $(n + 1)(2a + n) = 2N$ and $d$ is one of $n + 1$ and $(2a + n)$. We seek the smallest value of $N$ which has exactly 2019 odd divisors greater than 1, and so has exactly 2020 odd divisors. If $N = p_1^{e_1}p_2^{e_2} \cdots p_k^{e_k}$ where the $p_j$ are odd primes, then the number of odd divisors of $N$ is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$, so we want $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1) = 2020 = (2)(2)(5)(101)$. There are exactly 11 factorizations of 2020: (2020), (1010 \cdot 2), (505 \cdot 4), (505 \cdot 2 \cdot 2), (404 \cdot 5), (202 \cdot 10), (202 \cdot 5 \cdot 2), (101 \cdot 20), (101 \cdot 10 \cdot 2), (101 \cdot 5 \cdot 4), (101 \cdot 5 \cdot 2 \cdot 2).$ Each corresponds to an odd number $N$ with 2020 divisors. They are $p_1^{2019}p_2^{1009}p_2^1, p_1^{504}p_2^3, p_1^{403}p_2^2, p_1^{201}p_2^9, p_1^{100}p_2^{19}, p_1^{100}p_2^5p_3, p_1^{100}p_2^3p_3^2, p_1^{100}p_2^2p_3p_4$. With $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, and $p_4 = 11$, these eleven values of $N$ are in decreasing order, and the smallest is $3^{100} \cdot 5^4 \cdot 7 \cdot 11.$