1. A geometric progression.
The unique solution is $x = \frac{1}{3}$. We have
\[
\frac{8^{2x}}{4^{3x-1}} = \frac{16}{8^x}
\]
so
\[
\frac{2^{6x}}{2^{6x-2}} = \frac{2^4}{2^{6x}}; \quad 2^2 = 2^{4-6x}.
\]
Then $6x = 2$, and $x = \frac{1}{3}$.

2. A 2016 evaluation
We show that $x^3 + 1/x^3 = 2015\sqrt{2018}$. We have
\[
\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} = 2018,
\]
so $x + 1/x = \sqrt{2018}$. Then
\[
\left(x + \frac{1}{x}\right)^3 = 2018\sqrt{2018} = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + 3\sqrt{2018},
\]
and $x^3 + 1/x^3 = 2018\sqrt{2018} - 3\sqrt{2018} = 2015\sqrt{2018}$.

3. Some cubic polynomials.
They are $x^3, x^3 - ax^2$ for arbitrary $a \neq 0$, and $x^3 + x^2 - x - 1$.

From $x^3 - ax^2 + bx - c = (x-a)(x-b)(x-c)$ we have
\[
a + b + c = a, \quad (1)
\]
\[
ab + ac + bc = b, \quad (2)
\]
and
\[
abc = c. \quad (3)
\]
From (1) we see that $b + c = 0$, so (2) reduces to $bc = b$, which implies that $b = 0$ or $c = 1$. If $b = 0$, then $c = 0$ as well, and all three conditions are satisfied. This gives us $x^3$ and $x^3 - ax^2$ for arbitrary $a \neq 0$. If $b \neq 0$ then $c = 1$, and (1) implies $b = -1$ and (3) implies $a = -1$. This gives us the solution $x^3 + x^2 - x - 1$. It is easy to check that each of $x^3, x^3 - ax^2$ and $x^3 + x^2 - x - 1$ satisfies the specified condition.
4. A sum of squared sines.

The sum is 45.5. Pair the terms up as follows:

\[ S = \sin^2 0^\circ + \sin^2 90^\circ \\
+ \sin^2 1^\circ + \sin^2 89^\circ \\
+ \sin^2 2^\circ + \sin^2 88^\circ \\
+ \cdots \\
+ \sin^2 44^\circ + \sin^2 46^\circ \\
+ \sin^2 45^\circ. \]

Because \( \sin^2 (90^\circ - \theta) = \cos^2 \theta \), each of the pairs \( \sin^2 n^\circ + \sin^2 (90 - n)^\circ = \sin^2 n^\circ + \cos^2 n^\circ = 1 \) for \( 0 \leq n \leq 44 \). Thus we have

\[ S = 45 + \sin^2 45^\circ = 45 + \frac{1}{2}. \]

5. Lots of powers of 2.

Because \( |1 - \sqrt{3}| < 1 \), we have \( 0 < (1 - \sqrt{3})^{2n} < 1 \) for all positive integers \( n \). Note that \( (1 + \sqrt{3})^{2n} = r + s\sqrt{3} \) for some integers \( r \) and \( s \), and that \( (1 - \sqrt{3})^{2n} = r - s\sqrt{3} \). Thus

\[ (1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n} = 2r = \lceil (1 + \sqrt{3})^{2n} \rceil. \]

Now, \( (1 \pm \sqrt{3})^2/2 = 2 \pm \sqrt{3} \), so \( (1 \pm \sqrt{3})^{2n}/2^n = (2 \pm \sqrt{3})^n = t \pm u\sqrt{3} \) for some integers \( t \) and \( u \). Then

\[ \frac{\lceil (1 + \sqrt{3})^{2n} \rceil}{2^n} = \frac{2r}{2^n} = \frac{(1 + \sqrt{3})^{2n}}{2^n} + \frac{(1 - \sqrt{3})^{2n}}{2^n} = 2t, \]

showing that \( \lceil (1 + \sqrt{3})^{2n} \rceil / 2^{n+1} = t \), an integer.


The side lengths are \( 24/\sqrt{15}, 16/\sqrt{15} \) and \( 32/\sqrt{15} \). Let \( a, b, c \) be the sides corresponding to altitudes 4, 6 and 3, respectively. Then \( 4a = 6b = 3c \), each being twice the area of the triangle. The area of the triangle is also given by \( \sqrt{s(s - a)(s - b)(s - c)} \), where

\[ s = \frac{1}{2}(a + b + c) = \frac{1}{2} \left( a + \frac{2}{3}a + \frac{4}{3}a \right) = \frac{3}{2}a. \]

Equating the two expressions for the area we have

\[ 2a = \sqrt{\frac{3}{2}a \left( \frac{1}{2}a \right) \left( \frac{5}{6}a \right) \left( \frac{1}{6}a \right)} = \frac{a^2}{12} \sqrt{15}. \]

Thus \( a = 24/\sqrt{15}, b = 2a/3 = 16/\sqrt{15} \), and \( c = 4a/3 = 32/\sqrt{15} \).
7. Five triangles of equal area.

We show that $[FD = 8]$. Triangle $ADC$ is $4/5$ of the area of $ABC$ and they have the same altitude from the base along $AB$, so $AD = (4/5)(AB) = 24$. Similarly triangle $FED$ has $1/3$ the area of $AED$ and they have the same altitude from the base on $AD$. Therefore $FD = (1/3)(AD) = (1/3)(24) = 8$.

8. Divisible by $2^{2016}$.

Yes, there is such an integer. We show, by induction, that for every positive integer $n$, there is an $n$-digit integer $N$, each digit of which is 6 or 7, which is divisible by $2^n$. With $n = 1$ we take $N = 6$. We now show that if $N$ is a $k$-digit number, each digit 6 or 7, which is divisible by $2^k$, then one of the two numbers obtained by putting a 6 or a 7 in front of $N$ will be divisible by $2^{k+1}$. For example, from 6 we go to 76, divisible by 4, to 776, divisible by 8.

So, let $N$ be a $k$-digit number, each digit 6 or 7, and divisible by $2^k$. We may write $N = 2^k \cdot r$. The integer obtained by putting 6 or 7 in front of $N$ is $2^k \cdot r + 6 \cdot 10^k$ or $2^k \cdot r + 7 \cdot 10^k$, respectively. If $r$ is even, then $2^k \cdot r + 6 \cdot 10^k$ is divisible by $2^{k+1}$. If $r$ is odd, then $2^k \cdot r + 7 \cdot 10^k = 2^k \cdot r + 2^k \cdot 7 \cdot 5^k = 2^k(r + 7 \cdot 5^k)$, which is divisible by $2^{k+1}$ because $r + 7 \cdot 5^k$ is even. By induction, the claim is proved.

9. All terms integers?

We show that every term is an integer. Clear the recursion of fractions to get $a_n a_{n-2} = a_{n-1}^2 + 13$, and then also $a_{n+1} a_{n-1} = a_n^2 + 13$. Subtracting the first of these from the second gives $a_{n+1} a_{n-1} - a_n a_{n-2} = a_n^2 - a_{n-1}^2$, which we rewrite $a_n^2 + a_n a_{n-2} = a_n^2 + a_{n-1} a_{n+1}$. With a change of notation,

$$a_k(a_k + a_{k-2}) = a_{k-1}(a_{k-1} + a_{k+1}).$$

Consider this last equation now for $3 \leq k \leq n$:

$$a_3(a_3 + a_1) = a_2(a_2 + a_4)$$
$$a_4(a_4 + a_2) = a_3(a_3 + a_5)$$
$$a_5(a_5 + a_3) = a_4(a_4 + a_6)$$

$$\vdots$$

$$a_{n-1}(a_{n-1} + a_{n-3}) = a_{n-2}(a_{n-2} + a_n)$$
$$a_n(a_n + a_{n-2}) = a_{n-1}(a_{n-1} + a_{n+1}).$$
Equating the product of the left-hand members with the product of the right-hand members we obtain
\[(a_3a_4 \cdots a_n)(a_3 + a_1)(a_4 + a_2) \cdots (a_{n-1} + a_3)(a_n + a_{n-2}) = (a_2a_3 \cdots a_{n-1})(a_2 + a_4)(a_3 + a_5) \cdots (a_{n-2} + a_n)(a_{n-1} + a_1).\]
Canceling common factors leaves
\[a_n(a_3 + a_1) = a_2(a_{n-1} + a_{n+1}).\]
The initial terms are \(a_1 = 1\), \(a_2 = 7\) and \(a_3 = 62\), so \(63a_n = 7a_{n-1} + 7a_{n+1}\). Thus, \(a_{n+1} = 9a_n - a_{n-1}\), from which it is clear that all terms are integers.

10. An inequality.

We have
\[a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)((a - b)^2 + ab) \geq (a + b)(ab).\]
Similarly, \(b^3 + c^3 \geq (b + c)(bc)\) and \(c^3 + a^3 \geq (c + a)(ca)\). Adding these three inequalities, we obtain
\[2(a^3 + b^3 + c^3) \geq a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2.\]
Then
\[a^3 + b^3 + c^3 \geq a^2\left(\frac{b + c}{2}\right) + b^2\left(\frac{a + c}{2}\right) + c^2\left(\frac{a + b}{2}\right) \geq (a^2)\sqrt{bc} + (b^2)\sqrt{ac} + (c^2)\sqrt{ab},\]
where we have used the AM,GM inequality in the last step.