Each problem number is followed by an 11-tuple \((a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)\), where \(a_k\) is the number of teams that scored \(k\) points on the problem.

**1. Units digit of \(S_{2011}\).** \((28,1,1,0,1,0,9,2,5,5,24)\)

The units digit of \(S_{2011}\) is \(1\). We show this by observing that \(S_1 = 1\) has units digit 1, and that \(S_{n+10}\) always has the same units digit as does \(S_n\). Modulo 10, the sequence of squares of positive odd integers begins \((1, 9, 5, 9, 1)\), after which this cycle repeats itself, because \((n + 10)^2 \equiv n^2 \pmod{10}\). Then modulo 10, \(S_{n+10} - S_n\) will always be the sum of 10 consecutive terms in this sequence, which will be

\[1 + 9 + 5 + 9 + 1 + 9 + 5 + 9 + 1 = 0,\]

wherever in the sequence one begins. Thus \(S_{n+10} \equiv S_n \pmod{10}\).

**2. Area of a square.** \((60,3,0,2,0,2,1,0,1,0,7)\)

The area is \(\frac{8}{9}S\). The legs of the right triangle have length \(2\sqrt{S}\), so the hypotenuse is \(2\sqrt{2S}\). Then the second inscribed square has side \(\frac{2\sqrt{2S}}{3}\), because the smaller triangles at \(A\) and \(C\) are also isosceles right triangles. Thus the area of the second inscribed square is \(\left(\frac{2\sqrt{2S}}{3}\right)^2 = \frac{8S}{9}\).

**3 Speshul integers.** \((31,0,0,2,0,0,4,7,2,0,30)\)

Well, of course, every positive integer is speshul. Here are two different ways to get the \(m\) and \(n\) we need.

(1) To make \(mn + 1 = km + kn\), start by taking \(m = k + 1\). Then we want \(kn + n + 1 = km + kn\); i.e., \(n + 1 = km\), so \(n = km - 1\) does it. Thus: \(m = k + 1, n = km - 1 = k^2 + k - 1\).

(2) Take \(m = 2k - 1, n = 2k + 1\). Then

\[
\frac{mn + 1}{m + n} = \frac{4k^2}{4k} = k.
\]

**4. Divisor and remainder.** \((20,0,2,1,1,1,2,0,0,9,40)\)

The unique such pair is \((d, r) = (71, 52)\). From \(904 = ad + r\), \(1259 = bd + r\) and \(2040 = cd + r\) we obtain on subtraction,

\[355 = (b - a)d \quad \text{and} \quad 781 = (c - b)d.\]

Now, 781 = (2)(355) + 71, and both 781 and 355 are multiples of \(d\), so 71 is a multiple of \(d\). But 71 is prime, so \(d\) can only be 71 (the condition \(0 < r < d\) requires \(d > 1\)), and then \(904 = (12)(71) + 52\) shows that \(r\) must be 52. One checks that \(1259 = (17)(71) + 52\) and \(2040 = (28)(71) + 52\), so \((d, r) = (71, 52)\) is the unique solution.
5. Bigger than $2011^2$? \((18,0,1,0,2,0,2,4,5,6,38)\)

Such numbers do exist. Let \(f(x) = x^2 + \sqrt{2011-x}\) for \(0 \leq x \leq 2011\). It suffices to show that there exists \(x \in [0, 2011]\) with \(f(x) > 2011^2\). We note that \(f(2011) = 2011^2\), so it suffices to show that \(f'(x)\) is negative throughout an interval \((2011 - \epsilon, 2011)\) for some \(\epsilon > 0\). Well, \(f'(x) = 2x - 1/(2\sqrt{2011-x})\), and for \(0 \leq x \leq 2011\), \(2x < 4022\). But \(1/(2\sqrt{2011-x}) > 4022\) when \(1/\sqrt{2011-x} > 8044\); i.e., when \(2011 - x < 1/8044^2\). Thus, when

\[
2011 - \frac{1}{8044^2} < x < 2011,
\]

\(f'(x) < 0\), so \(f(x)\) decreases in this interval to the value \(2011^2\) at \(x = 2011\), and hence \(f(x) > 2011^2\) throughout this interval.

6. From 11 to 2011. \((38,1,1,1,0,0,1,2,0,0,32)\)

Yes, it can be done in many ways (infinitely many, in fact). A key observation is that with any \(n > 1\), \(n = (n-1) + 1\) can be replaced by \((n-1) \cdot 1 = n - 1\), so if we obtain any integer larger than 2011, we can step down from there to 2011 in steps of 1. Thus, one solution is

\[
11 = 6 + 5 \rightarrow 30 = 15 + 15 \rightarrow 225 = 10 + 215 \rightarrow 2150 = 2149 + 1 \rightarrow 2149,
\]
e tc., down to 2011.

7. Not a perfect square. \((11,0,0,0,0,1,0,2,2,1,59)\)

We show that \((n+5)^2 < a_n < (n+6)^2\) for all \(n\), and the claim follows. Using the facts that \(4 < \sqrt{19} < 5\) and \(9 < \sqrt{99} < 10\) we have

\[
a_n = \lfloor n^2 + 2\sqrt{19n + 19} + 2n + \sqrt{99} \rfloor
\leq \lfloor n^2 + 12n + 29 \rfloor
= \lfloor (n+6)^2 - 7 \rfloor
< (n+6)^2,
\]

and

\[
a_n \geq \lfloor n^2 + 10n + 28 \rfloor
= \lfloor (n+5)^2 + 3 \rfloor
> (n+5)^2.
\]
8. $P(11) = 2011$. (35,1,6,0,1,0,0,0,0,0,33)

We show that $P(6) = 441$. From the fact that $2011 = P(11) \geq a_n \cdot 11^n \geq 11^n$ we know that $n \leq 3$ because $11^3 = 1331 < 2011 < 11^4$. Also, $2011 < 2 \cdot 11^3$, so $a_3 \leq 1$, and thus if $n = 3$ then $a_3 = 1$. We show that $n = 3$. If $n \leq 2$, then $P(x) = a_0 + a_1 x + a_2 x^2 \leq 10 + 10x + 10x^2$, and $2011 = P(11) \leq 10 + 110 + 1210 < 2011$, a contradiction. Thus $n = 3$, $a_3 = 1$, and $P(x) = a_0 + a_1 x + a_2 x^2 + x^3$. Then $2011 = P(11) = a_0 + a_1 \cdot 11 + a_2 \cdot 11^2 + 1331$, whence $a_0 + a_1 \cdot 11 + a_2 \cdot 11^2 = 2011 - 1331 = 680$.

Because $0 \leq a_0 + a_1 \cdot 11 \leq 10 + 110 = 120$, we have $680 \geq a_2 \cdot 11^2 \geq 680 - 120 = 560$, and therefore $6 > 680/121 \geq a_2 \geq 560/121 > 4$. Hence $a_2 = 5$, and

$$2011 = a_0 + a_1 \cdot 11 + 5 \cdot 121 + 1331 = a_0 + 11a_1 + 1936,$$

so $a_0 + 11a_1 = 75$. The constraints on $a_0$ and $a_1$ then determine that $a_1 = 6$ and $a_0 = 9$. Thus $P(x) = 9 + 6x + 5x^2 + x^3$. Note that this also satisfies $P(1) = 21$ (which was given but not used for the solution). Finally, then, $P(6) = 9 + 36 + 180 + 216 = 441$.

9. Fractional part of $\sqrt{n}$. (8,0,2,0,0,0,0,0,0,0,66)

Let $r$ be the integer such that $r - 1 < \sqrt{n} < r$. If $r = 2$, then $n = 2$ or 3 and one can easily verify directly that the required inequality holds. So we assume that $r \geq 3$, in which case $(r - 1)^2 \geq r$, and hence $n > r$. Let $k = r^2 - n$. Then $k > 0$ so $k \geq 1$, and we have

$$x_n = \sqrt{n} - \lfloor \sqrt{n} \rfloor = \sqrt{n} - (r - 1),$$

and then

$$x_n + \frac{1}{2n} = \sqrt{n} - r + 1 + \frac{1}{2n}$$

$$= \sqrt{r^2 - k} - r + 1 + \frac{1}{2n}$$

$$\leq \sqrt{r^2 - 1} - r + 1 + \frac{1}{2n}$$

$$\leq \sqrt{r^2 - 1} - r + 1 + \frac{1}{2r}$$

$$= 1 - \left( r - \sqrt{r^2 - 1} - \frac{1}{2r} \right)$$

$$= 1 - \left( \frac{1}{r + \sqrt{r^2 - 1}} - \frac{1}{2r} \right) < 1,$$

q.e.d.
10. Numerator divisible by 2011. \((3,1,0,0,0,0,0,0,0,0,72)\)

We rewrite
\[
\frac{p}{q} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1339} + \frac{1}{1340}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1340}\right)
\]
\[
= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1339} + \frac{1}{1340}\right) - \left(\frac{1}{2} + \cdots + \frac{1}{670}\right)
\]
\[
= \frac{1}{671} + \frac{1}{672} + \cdots + \frac{1}{1340},
\]

Now group these terms in pairs as follows:
\[
\frac{p}{q} = \left(\frac{1}{671} + \frac{1}{1340}\right) + \left(\frac{1}{672} + \frac{1}{1339}\right) + \cdots + \left(\frac{1}{1005} + \frac{1}{1006}\right)
\]
\[
= \frac{1340 + 671}{671 \cdot 1340} + \frac{1339 + 672}{672 \cdot 1339} + \cdots + \frac{1006 + 1005}{1005 \cdot 1006}
\]
\[
= 2011\left(\frac{1}{671 \cdot 1340} + \frac{1}{672 \cdot 1339} + \cdots + \frac{1}{1005 \cdot 1006}\right)
\]
\[
= 2011\frac{r}{s},
\]

for some integers \(r\) and \(s\), where all prime factors of \(s\) are smaller than 2011. We now have \(2011rq = ps\), 2011 is prime, and 2011 does not divide \(s\). Therefore \(2011|p\), q.e.d.