Each problem number is followed by an 11-tuple \((a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)\), where \(a_k\) is the number of teams that scored \(k\) points on the problem.

1. **Angle relationship.** (57,1,2,0,0,2,2,1)

   \[ a = d - b - c. \]

   For, \(a + b + c + e + f = 180\) and \(e + f = 180 - d\), so \(a + b + c + (180 - d) = 180\), and thus \(a + b + c - d = 0; a = d - b - c.\)

2. **System of equations.** (40,0,0,1,0,1,8,0,9,1,5)

   The only solution is \((\frac{32}{5}, -\frac{14}{5})\). If \(x < 0\) the first equation implies \(y = 10\). But when \(y = 10\), the second equation implies \(x = 12\), so there are no solutions with \(x < 0\). If \(y \geq 0\), the second equation reduces to \(x = 12\). But with \(x = 12\) in the first equation we get \(y < 0\), so there are no solutions with \(y \geq 0\). This leaves only the fourth quadrant. If \(x \geq 0\) and \(y < 0\) the equations simplify to \(2x + y = 10\) and \(x - 2y = 12\), with the unique solution \(x = \frac{32}{5}, y = -\frac{14}{5}\).

**GRAPH-AIDED SOLUTION**

The graph of the first equation is the horizontal line \(y = 10\) when \(x < 0\) and the half-line of slope \(-2\) connected to this when \(x \geq 0\). The graph of the second equation consists of the vertical line \(x = 12\) when \(y \geq 0\) and the half-line with slope \(\frac{1}{2}\) connected to it when \(y < 0\). Where the graphs intersect, the equations are \(y = 10 - 2x\) and \(x = 12 + 2y\), with solution \((\frac{32}{5}, -\frac{14}{5})\).
3. Coefficient of $x^9$. \((39,0,7,2,0,1,0,1,0,1,14)\)

It is \(-112640\). The expansion may be written

$$\sum_{k=0}^{12} \binom{12}{k} (2x^2)^k (-x^{-3})^{12-k} = \sum_{k=0}^{12} \binom{12}{k} (-2)^k x^{5k-36}.$$  

Now, $5k - 36 = 9$ when $k = 9$, and the corresponding coefficient is

$$-\binom{12}{9} 2^9 = -\frac{12 \cdot 11 \cdot 10}{2 \cdot 3} 2^9 = -11 \cdot 10 \cdot 2^{10} = -112640.$$  

4. Final digit. \((42,0,1,2,2,2,9,1,3,1,2)\)

The last digit is 3. The final digits of 7, $7^2$, $7^3$ and $7^4$ are, respectively, 7, 9, 3, 1. Thereafter the cycle (7,9,3,1) repeats. Therefore we need only know $7^7$ modulo 4 to answer the question. But $7 \equiv 3 \mod 4$ and $7^2 \equiv 1 \mod 4$, so $7^7 \equiv 7 \equiv 3 \mod 4$. Thus $7^7$ has the same final digit as $7^3$, namely, 3.

5. A pair of integrals. \((27,0,1,0,3,2,0,0,3,2,27)\)

$$F(n) = I_n + I_{n+2} = \frac{1}{n+1}.$$  

We have

$$I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n+2} x) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x dx$$

$$= \int_0^1 u^n du = \frac{1}{n+1},$$  

where in the next to last step we use the substitution $u = \tan x$.  

6. Special integers. (12,3,6,1,0,5,0,2,7,11,18)

All positive integers \( n \neq 2 \) are special. We have already seen that 3 is. Suppose \( k \) is special, \( k \geq 3 \), and \( 1 = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} \), with \( d_1 < d_2 < \cdots < d_k \). Now,

\[
\frac{1}{d_k} = \frac{d_k + 1}{d_k(d_k + 1)} = \frac{1}{d_k} + \frac{1}{d_k(d_k + 1)},
\]

so upon replacing \( \frac{1}{d_k} \) by \( \frac{1}{d_k} + \frac{1}{d_k(d_k + 1)} \), we see that \( k + 1 \) is special. It follows by induction that all integers \( n \geq 3 \) are special, and it is evident that 1 is. To show that 2 is not, consider any sum \( \frac{1}{a} + \frac{1}{b} \), where \( a < b \) and \( a \) and \( b \) are integers. If \( a = 1 \) the sum is larger than 1. If \( b > a \geq 2 \), then the sum is less than 1, so there do not exist distinct integers \( a, b \) with \( \frac{1}{a} + \frac{1}{b} = 1 \).

7. Card count. (37,0,2,1,7,8,1,0,1,0,8)

The sum is 2652. Let the cards of the deck be numbered 1, 2, \ldots, 52. Let \( a_i \) be the position of card \( i \) in the lower deck, and \( b_i \) the position in the whole stack of card \( i \) in the upper deck. Then \( (a_1, a_2, \ldots, a_{52}) \) is a permutation of \( (1, 2, \ldots, 52) \) and \( (b_1, b_2, \ldots, b_{52}) \) is a permutation of \( (53, 54, \ldots, 104) \). Then the number of cards between card \( i \) of the upper deck and card \( i \) of the lower deck is \( b_i - a_i - 1 \), and the sum of these numbers is

\[
\sum_{i=1}^{52} (b_i - a_i - 1) = \sum_{i=1}^{52} b_i - \sum_{i=1}^{52} a_i - 52
\]

\[
= (53 + 54 + \cdots + 104) - (1 + 2 + \cdots + 52) - 52
\]

\[
= 26(53 + 104) - 26(1 + 52) - 52
\]

\[
= (26)(102) = 2652.
\]

**Alternate Solution**

First do the count with both decks in their natural order:

\( (1, 2, \ldots, 52)(53, 54, \ldots, 104) \)

In this position, each card is separated from its counterpart in the other deck by 51 intermediate cards, so the count in question is \( (51)(52) = 2652 \). Now if two cards are transposed within one deck, one of the distances increases by some integer \( k \) and the other decreases by \( k \). By a suitable sequence of transpositions each of the decks may be arranged to an arbitrary order, and the count in question is unchanged.
8. Sum the series. \((10,0,0,0,1,0,2,1,0,1,50)\)

Here one should try to find a pattern in the partial sums, and prove it by induction. The first few partial sums are

\[
S_1 = \frac{1}{3}, \quad S_2 = \frac{7}{3 \cdot 5}, \quad S_3 = \frac{52}{3 \cdot 5 \cdot 7}.
\]

The pattern can be seen most readily by comparing with the asserted limit, 1/2. Look at the differences 1/2 – \(S_n\). Thus we find

\[
S_1 = \frac{1}{2} \left(1 - \frac{1}{3}\right), \quad S_2 = \frac{1}{2} \left(1 - \frac{1}{3 \cdot 5}\right) \quad S_3 = \frac{1}{2} \left(1 - \frac{1}{3 \cdot 5 \cdot 7}\right),
\]

suggesting the formula

\[
S_n = \frac{1}{2} \left(1 - \frac{1}{3 \cdot 5 \cdots (2n + 1)}\right), \quad (1)
\]

This is easily proved by induction. We have it for \(n = 1\). Suppose it holds for a particular \(n\). Then

\[
S_{n+1} = S_n + \frac{n + 1}{3 \cdot 5 \cdots (2n + 1)(2n + 3)}
\]

\[
= \frac{1}{2} \left(1 - \frac{1}{3 \cdot 5 \cdots (2n + 1)}\right) + \frac{n + 1}{3 \cdot 5 \cdots (2n + 1)(2n + 3)}
\]

\[
= \frac{1}{2} \left(1 - \frac{2n + 3}{3 \cdot 5 \cdots (2n + 1)(2n + 3)} + \frac{2n + 2}{3 \cdot 5 \cdots (2n + 1)(2n + 3)}\right)
\]

\[
= \frac{1}{2} \left(1 - \frac{1}{3 \cdot 5 \cdots (2n + 1)(2n + 3)}\right).
\]

By induction, (1) holds for all \(n\), and it follows at once that \(\lim S_n = 1/2\).

9. Product of secants. \((5,0,0,0,0,0,0,0,0,0,60)\)

Let \(P = \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7}\) and \(Q = \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7}\). Then

\[
8PQ = \left(2 \sin \frac{\pi}{7} \cos \frac{\pi}{7}\right) \left(2 \sin \frac{2\pi}{7} \cos \frac{2\pi}{7}\right) \left(2 \sin \frac{3\pi}{7} \cos \frac{3\pi}{7}\right)
\]

\[
= \sin \frac{\pi}{7} \sin \frac{4\pi}{7} \sin \frac{6\pi}{7}
\]

\[
= \sin \frac{\pi}{7} \sin \frac{3\pi}{7} \sin \frac{\pi}{7} = Q
\]

It is clear that \(Q \neq 0\), so we have \(8P = 1\), and

\[
\sec \frac{\pi}{7} \sec \frac{2\pi}{7} \sec \frac{3\pi}{7} = \frac{1}{P} = 8.
\]
10. Powers of an irrational number. \((7,1,0,0,2,1,1,2,2,11,38)\)

We have \(r = \sqrt{3} + \sqrt{2}\) and \(r^{-1} = \sqrt{3} - \sqrt{2}\). Let \(s = r^{-1}\). Then \(r^2 = 5 + 2\sqrt{6}\), \(r^{-2} = s^2 = 5 - 2\sqrt{6}\), so we see that (i) and (ii) hold with \(n = 1\) and \(a_1 = 2\). Also, (i) and (ii) hold for \(n = 0\) with \(a_0 = 0\). We will do an induction with the aid of a recursion. Note that \((r^2 - 5)^2 = 24\), which implies that \(r^4 = 10r^2 - 1\). It follows that for every \(n\),

\[
 r^{2n+2} = 10r^{2n} - r^{2n-2}. \tag{*}
\]

Moreover, \((s^2 - 5)^2 = 24\) also, so

\[
 s^{2n+2} = 10s^{2n} - s^{2n-2}. \tag{**}
\]

We can now prove (i) by induction on \(n\). We have it for \(n = 0\) and 1. Suppose that

\[
 r^{2n-2} + s^{2n-2} = 4a_{n-1} + 2, \quad \text{and} \quad r^{2n} + s^{2n} = 4a_n + 2
\]

Then from (*) and (**) we have

\[
 r^{2n+2} + s^{2n+2} = 10r^{2n} - r^{2n-2} + 10s^{2n} - s^{2n-2}
\]

\[
 = 10(r^{2n} + s^{2n}) - (r^{2n-2} + s^{2n-2})
\]

\[
 = 10(4a_n + 2) - (4a_{n-1} + 2)
\]

\[
 = 4a_{n+1} + 2,
\]

where \(a_{n+1} = 10a_n - a_{n-1} + 4\). By induction, (i) holds for all \(n\). Let \(x = r^{2n}\). Then from (i) we have \(x + x^{-1} = 4a_n + 2\), and

\[
 x^2 - (4a_n + 2)x + 1 = 0.
\]

Then

\[
 r^{2n} = x = \frac{4a_n + 2 + \sqrt{(4a_n + 2)^2 - 4}}{2}
\]

\[
 = 2a_n + 1 + 2\sqrt{a_n(a_n + 1)}
\]

\[
 = (\sqrt{a_n + 1} + \sqrt{a_n})^2,
\]

(where the + sign was used in the quadratic formula because \(r > 1\)). It follows that

\[
 r^n = \sqrt{a_n + 1} + \sqrt{a_n},
\]

proving (ii). (The other root of the quadratic is \(r^{-2n} = 2a_n + 1 - 2\sqrt{a_n(a_n + 1)}\).)