2018 Missouri Collegiate Mathematics Competition
Session I

1. Let f be a real-valued function such that for any real x,

(a) $f(10+x) = f(10-x)$.

(b) $f(20+x) = -f(20-x)$.

Prove that f is an odd and periodic function.

Solution.

Let $x = n - 10$ in (a) to get $f(n) = f(20-n)$. Also, let $x = n$ in (b) to get $f(20-n) = -f(20+n)$. So $f(n) = -f(20+n)$. Now let $x = n+10$ in (a) to get $f(n+20) = f(-n)$, which means $-f(n) = f(-n)$ and f is odd. Further, let $x = n-20$ in (b) to get $f(n) = -f(40-n) = f(n-40)$ since we now know f is odd. So, f is also periodic.

2. Let $a > 0$. Suppose f is a continuous function on $[0, a]$ such that $f(x)\, f(a-x) = 1$. Evaluate

$$\int_0^a \frac{dx}{1+f(x)}.$$

Solution.

Setting $x = a - y$, we obtain

$$\begin{aligned} I &= \int_0^a \frac{dx}{1+f(x)} = \int_0^a \frac{dy}{1+f(a-y)} \\ &= \int_0^a \frac{f(y)\, dy}{f(y)+1} = \int_0^a \left[1 - \frac{1}{1+f(y)}\right] dy \\ &= a - I \end{aligned}$$

so that

$$I = \frac{a}{2}.$$

Hardy and Williams, The Green Book of Mathematical Problems, p. 6 #26

3. In a theoretical version of the Canadian lottery "Lotto 6–49", a ticket consists of six distinct integers chosen by the player from 1 to 49 (inclusive). A t-prize is awarded for any ticket having t or more numbers in common with a designated "winning" ticket. Denote by $f(t)$ the smallest number of tickets required to be certain of

winning a t-prize. Clearly $f(1) = 8$ and $f(6) = \binom{49}{6}$. Show that $f(2) \leq 19$.

Solution.

Let a–b denote the set $\{a, a+1, \ldots, b\}$.

To show that $f(2) \leq 19$, partition 1–49 into five non-empty disjoint subsets; 1–10, 11–20, 21–30, 31–40, 41–49. Since the "winning" ticket has six distinct numbers in 1–49, one of these subsets must contain at least two "winning" numbers. First, suppose that 1–10 contains at least two "winning" numbers. Now, every pair of distinct numbers in 1–10 appears on one of the four tickets

$$\begin{array}{rl} & 1,\ 2,\ 3,\ 4,\ 5,\ 6; \\ & 5,\ 6,\ 7,\ 8,\ 9,\ 10; \\ & 1,\ 2,\ 7,\ 8,\ 9,\ 10; \\ \text{and} & 3,\ 4,\ 7,\ 8,\ 9,\ 10. \end{array}$$

Thus, if we have these four tickets, we are certain of winning a 2-prize. Similarly, if 11–20 contains at least two "winning" numbers, the four tickets

$$\begin{array}{rl} & 11,\ 12,\ 13,\ 14,\ 15,\ 16; \\ & 15,\ 16,\ 17,\ 18,\ 19,\ 20; \\ & 11,\ 12,\ 17,\ 18,\ 19,\ 20; \\ \text{and} & 13,\ 14,\ 17,\ 18,\ 19,\ 20 \end{array}$$

guarantee winning a 2-prize. The same pattern of four tickets will assure winning a 2-prize if 21–30 (or 31–40) contains at least two "winning" numbers. Finally, if 41–49 contains at least two "winning" numbers, the three tickets

$$\begin{array}{rl} & 41,\ 42,\ 43,\ 44,\ 45,\ 46; \\ & 44,\ 45,\ 46,\ 47,\ 48,\ 49; \\ \text{and} & 41,\ 42,\ 43,\ 47,\ 48,\ 49 \end{array}$$

guarantee winning a 2-prize.

Therefore, we are certain of winning a 2-prize if we have the above $4 \cdot 4 + 3 = 19$ tickets. Thus, $f(2) \leq 19$.

Problem 1338 from Crux Mathematicorum, Vol. 14, No. 3 (April 1988), p. 110. Proposed by Jean Dooyen, Université Libre de Bruxelles, Brussels, Belgium; J. Chris Fisher, University of Regina, Regina, Saskatchewan; and Gunter Kist, Technische Universitat, Munich, Federal Republic of Germany.

4. Find all real numbers x that satisfy $6^x + 1 = 8^x - 27^{x-1}$. Prove your solution set contains all solutions.

Solution.

Let $f(x) = 6^x + 27^{x-1} + 1$ and $g(x) = 8^x$. Then the problem is equivalent to finding all solutions of $f(x) = g(x)$.

For $x < 0$, $f(x) > 1$ and $g(x) < 1$, so there are no negative solutions. Also, $f(0) > g(0)$.

Experimenting leads to $f(1) = g(1) = 8$ and $f(2) = g(2) = 64$, so $x = 1$ and $x = 2$ are solutions. We now prove these are the only solutions.

$f'(x) = 6^x \log 6 + 27^{x-1} \log 27$ and $f''(x) = 6^x(\log 6)^2 + 27^{x-1}(\log 27)^2$.

We consider $\log(f(x))$ and $\log(g(x))$, noting that the equation $\log(f(x)) = \log(g(x))$ has the same solutions as $f(x) = g(x)$. Now, $\log(g(x)) = x \log 8$, a straight line, so if we can show that the graph of $\log(f(x))$ is always concave up (or always concave down), then the graphs of the two log functions can only intersect at the two points already found, when $x = 1$ and $x = 2$. So, consider the second derivative of $\log(f(x))$.

$$\frac{d^2}{dx^2}\log(f(x)) = \frac{f(x)f''(x) - (f'(x))^2}{(f(x))^2}$$

After some algebra, we see that the numerator can be written as

$$6^x(\log 6)^2 + 27^{x-1}(\log 27)^2 + 6^x 27^{x-1}(\log 27 - \log 6)^2.$$

Thus, $\dfrac{d^2}{dx^2}\log(f(x)) > 0$ for all x, making the graph of $\log(f(x))$ always concave up, and so $x = 1$ and $x = 2$ are the only solutions.

This problem appeared in Mathematical Reflections by Titu Andreescu.

5. Let P_1, P_2, and P_3 be three points on the parabola $y = x^2$, and let ℓ_1, ℓ_2, and ℓ_3 be the tangent lines to the parabola at these points. The tangent lines intersect pairwise in three points. Denote the intersection of ℓ_1 and ℓ_2 by Q_{12}, the intersection of ℓ_1 and ℓ_3 by Q_{13}, and the intersection of ℓ_2 and ℓ_3 by Q_{23}. Find the ratio of the area of triangle $P_1P_2P_3$ to the area of triangle $Q_{12}Q_{13}Q_{23}$.

Solution.

Let $P_1 = (a, a^2)$, $P_2 = (b, b^2)$, and $P_3 = (c, c^2)$ with $a \neq b \neq c$. The equations of the tangent lines are:

$$\ell_1 : \ y = 2ax - a^2, \qquad \ell_2 : \ y = 2bx - b^2, \quad \text{and} \quad \ell_3 : \ y = 2cx - c^2.$$

Thus,

$$Q_{12} = \left(\frac{a+b}{2}, ab\right), \qquad Q_{13} = \left(\frac{a+c}{2}, ac\right), \quad \text{and} \quad Q_{23} = \left(\frac{b+c}{2}, bc\right).$$

Recall that the magnitude of the cross product of two vectors gives the area of the parallelogram defined by the two vectors, or twice the area of the triangle formed by the vectors. Thus,

$$\begin{aligned} 2 * Area(P_1P_2P_3) &= \left|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right| \\ &= \left\| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ b-a & b^2-a^2 & 0 \\ c-a & c^2-a^2 & 0 \end{array} \right\| \\ &= \left|(b-a)(c^2-a^2)-(c-a)(b^2-a^2)\right| \\ &= |(b-a)(c-a)(c-b)| \end{aligned}$$

Similarly,

$$\begin{aligned} 2 * Area(Q_{12}Q_{13}Q_{23}) &= \left|\overrightarrow{Q_{12}Q_{13}} \times \overrightarrow{Q_{12}Q_{23}}\right| \\ &= \left\| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{2}(c-b) & a(c-b) & 0 \\ \frac{1}{2}(c-a) & b(c-a) & 0 \end{array} \right\| \\ &= \frac{1}{2}\left|(c-b)(c-a)b-((c-b)(c-a)a\right| \\ &= \frac{1}{2}\left|(b-a)(c-a)(c-b)\right| \end{aligned}$$

So we have

$$\frac{Area(P_1P_2P_3)}{Area(Q_{12}Q_{13}Q_{23})} = 2.$$

This problem is from Mathematics Problems for High School by Gh. Călugăriţa and V. Mangu.

2018 Missouri Collegiate Mathematics Competition
Session II

1. For each positive integer n, let $a_n = \sum_{k=0}^{n} \frac{\pi^{4k}}{(4k+1)!}$ and $b_n = \sum_{k=0}^{n} \frac{\pi^{4k}}{(4k+3)!}$. Evaluate $\lim_{n\to\infty} \frac{a_n}{b_n}$.

Solution.

Clearly $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ exist. Thus we wish to compute the ratio A/B. Since the Taylor series expansion for sine is $\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, we see that $\pi A - \pi^3 B = \sin(\pi) = 0$. Thus $\pi A = \pi^3 B$ and so $A/B = \pi^2$.

This problem is essentially a Soviet Union Mathematical Olympiad problem from 1975.

2. Triangle ABC is a right triangle with right angle at C. Set $a = BC$, $b = CA$, and $c = AB$. Inscribe a circle inside triangle ABC, and inside the circle inscribe a triangle similar to triangle ABC. Repeat the process, obtaining an infinite sequence of circles and similar triangles. Find in terms of a, b, c the sum of the areas of the circles.

Solution.

Set $a_0 = a$, $b_0 = b$, and $c_0 = c$, and let a_k, b_k, c_k be the lengths of the corresponding sides in the k^{th} triangle. For $k = 0, 1, 2, \ldots$, let r_k denote the radius of circle k. Set $\lambda = \dfrac{a_0}{c_0}$, $\mu = \dfrac{b_0}{c_0}$. Then

$$\begin{aligned} r_k &= \frac{\text{area of } k^{th} \text{ triangle}}{\text{semiperimeter of } k^{th} \text{ triangle}} \\ &= \frac{a_k b_k}{a_k + b_k + c_k} \\ &= \frac{\lambda\mu c_k^2}{\lambda c_k + \mu c_k + c_k} \\ &= \frac{\lambda\mu c_k}{\lambda + \mu + 1}. \end{aligned}$$

On the other hand, when we inscribe the k^{th} triangle, the midpoint of its hypotenuse will be at the center of the $(k-1)^{st}$ circle. Thus,

$$c_k = 2r_{k-1}.$$

Therefore,

$$r_k = \frac{2\lambda\mu r_{k-1}}{\lambda + \mu + 1}.$$

Setting

$$d = \frac{2\lambda\mu}{\lambda + \mu + 1},$$

$$r_k = d^k r_0,$$

and the sum of the areas of the circles is

$$\begin{aligned} & \sum_{k=0}^{\infty} \pi r_k^2 \\ = \; & \pi \sum_{k=0}^{\infty} d^{2k} r_0^2 \\ = \; & \pi r_0^2 \sum_{k=0}^{\infty} \left(d^2\right)^k \\ = \; & \pi r_0^2 \left(\frac{1}{1-d^2}\right). \end{aligned}$$

We have

$$d = \frac{2ab}{c\,(a+b+c)} \cdot \frac{a+b-c}{a+b-c} = \frac{2ab\,(a+b-c)}{c\left[(a+b)^2 - c^2\right]} = \frac{2ab\,(a+b-c)}{2abc} = \frac{a+b-c}{c},$$

so the sum of the areas is

$$\begin{aligned} & \pi \left(\frac{ab}{a+b+c}\right)^2 \left[\frac{1}{1-\left(\frac{a+b-c}{c}\right)^2}\right] \\ = \; & \pi \left(\frac{ab}{a+b+c}\right)^2 \left[\frac{c^2}{c^2-(a+b-c)^2}\right] \\ = \; & \frac{\pi\,(abc)^2}{(a+b+c)^2\,(a+b)\,(2c-a-b)}. \end{aligned}$$

3. What is the coefficient of x^{2018} in

$$\begin{aligned} & (x+1)^3(x^2+1)^4(x^4+1)^5 \\ & \quad \times (x^8+1)(x^{16}+1)(x^{32}+1)(x^{64}+1)(x^{128}+1)(x^{256}+1)(x^{512}+1)(x^{1024}+1)? \end{aligned}$$

Solution.

Let $g(x) = (x+1)^2(x^2+1)^3(x^4+1)^4$, and let

$$\begin{aligned} h(x) &= (x+1)(x^2+1)(x^4+1)\dots(x^{256}+1)(x^{512}+1)(x^{1024}+1) \\ &= 1+x+x^2+x^3+x^4+\cdots+x^{2046}+x^{2047}, \end{aligned}$$

where the last equation follows from the unique expansion of a positive integer as a sum of powers of 2. If $f(x)$ is the polynomial in the problem statement, then $f(x) = g(x)h(x)$, so we want to know the coefficient of x^{2018} in the product $g(x)h(x)$. Writing $g(x) = b_{24}x^{24} + b_{23}x^{23} + \cdots + b_2x^2 + b_1x + b_0$, we see that the coefficient of x^{2018} in $g(x)h(x)$ equals $\sum_{i=0}^{24} b_i \cdot 1$ (since the coefficient of x^{2018-i} in $h(x)$ is 1). On the other hand, this sum is just $g(1) = 2^2 \cdot 2^3 \cdot 2^4 = 2^9 = 512$.

Similar to a Univ. of South Carolina practice problem for the American Regions Mathematics Competition.

4. Let P_2 denote the set of real polynomials of degree less than or equal to 2. Define the map $J : P_2 \to \mathbb{R}$ by

$$J(f) = \int_0^1 [f(x)]^2 \, dx.$$

Let $Q = \{f \in P_2 : f(1) = 1\}$. Show that J attains a minimum value on Q, and determine where the minimum occurs.

Solution.

For $f \in P_2$, write $f(x) = ax^2 + bx + c$. The condition $f(1) = 1$ can be written as $a + b + c = 1$. Thus, for $f \in Q$,

$$J(f) = \int_0^1 \left(ax^2 + bx + 1 - a - b\right)^2 \, dx.$$

Setting

$$G(a,b) = \int_0^1 \left(ax^2 + bx + 1 - a - b\right)^2 \, dx,$$

we have

$$\frac{\partial G(a,b)}{\partial a} = \int_0^1 2\left(ax^2 + bx + 1 - a - b\right)\left(x^2 - 1\right) \, dx = \frac{16}{15}a + \frac{5}{6}b - \frac{4}{3}$$

and

$$\frac{\partial G(a,b)}{\partial b} = \int_0^1 2\left(ax^2 + bx + 1 - a - b\right)(x - 1) \, dx = \frac{5}{6}a + \frac{2}{3}b - 1.$$

Thus, at a critical point, we have the system

$$\begin{cases} 0 = \frac{16}{15}a + \frac{5}{6}b - \frac{4}{3} = \frac{1}{30}(32a + 25b - 40) \\ 0 = \frac{5}{6}a + \frac{2}{3}b - 1 = \frac{1}{6}(5a + 4b - 6) \end{cases}.$$

Solving gives

$$a = \frac{10}{3},\ b = -\frac{8}{3},\ c = \frac{1}{3}$$

as the only critical point. Hence, if J attains a minimum, it must do so at this point. To see that J does attain a minimum, view J (restricted to Q) as a function of a and b. The graph of $z = J(a, b)$ is a quadric surface whose ab cross term can be eliminated by rotation about the z-axis. The result is an elliptic paraboloid, so it has a minimum.

de Souza and Silva, Berkeley Problems in Mathematics, 2nd ed., p. 33 #2.2.15

5. If A, B, C are angles of an acute triangle, prove that

$$(\tan A + \tan B + \tan C)^2 \geq (\sec A + 1)^2 + (\sec B + 1)^2 + (\sec C + 1)^2.$$

Solution.

Since A, B, C are acute, all the standard trigonometric functions of these angles are positive.

Note that

$$\begin{aligned}
&(\cos A - \cos B)^2 = \cos^2 A + \cos^2 B - 2\cos A \cos B \\
&= -\cos(B + C)\cos A - \cos(A + C)\cos B - 2\cos A \cos B \\
&= -\cos B \cos C \cos A + \sin B \sin C \cos A - \cos A \cos C \cos B \\
&\qquad + \sin A \sin C \cos B - 2\cos A \cos B \\
&= \cos A \cos B \cos C(\tan B \tan C + \tan A \tan C - 2\sec C - 2).
\end{aligned}$$

Therefore, since the left side is nonnegative,

$$\tan B \tan C + \tan A \tan C \geq 2\sec C + 2.$$

Similarly,

$$\tan A \tan B + \tan A \tan C \geq 2\sec A + 2$$

and

$$\tan A \tan B + \tan B \tan C \geq 2\sec B + 2.$$

Adding, we get

$$\tan A \tan B + \tan A \tan C + \tan B \tan C \geq \sec A + \sec B + \sec C + 3.$$

Hence

$$\begin{aligned}
(\tan A + \tan B + \tan C)^2 &= \tan^2 A + \tan^2 B + \tan^2 C \\
&\quad + 2(\tan A \tan B + \tan A \tan C + \tan B \tan C) \\
&\geq \tan^2 A + \tan^2 B + \tan^2 C + 2(\sec A + \sec B + \sec C) + 6 \\
&= \sec^2 A + \sec^2 B + \sec^2 C + 2(\sec A + \sec B + \sec C) + 3 \\
&= (\sec A + 1)^2 + (\sec B + 1)^2 + (\sec C + 1)^2.
\end{aligned}$$

Problem 1210 from Crux Mathematicorum - April 1988.