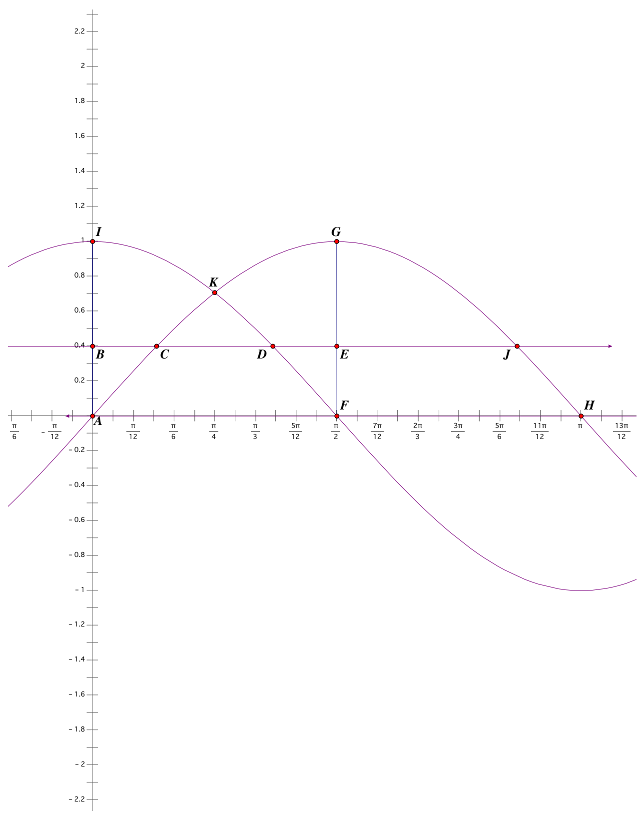
2016 Missouri Collegiate Mathematics Competition
Session I

1. Let $\mathcal{R}$ be the region in the first quadrant bounded by the x-axis and the graph of $y = \sin x$ from $x = 0$ to $x = \pi$. The graph of $y = \cos x$ divides $\mathcal{R}$ into two regions. Let $\mathcal{R}_1$ be the subregion of $\mathcal{R}$ bounded on the left by $y = \cos x$, on the right by $y = \sin x$, and below by the x-axis. Find a positive number b such that the line $y = b$ divides $\mathcal{R}_1$ into two regions of equal area.

Solution.



In the figure, the region $\mathcal{R}_1$ is $KDFHJGK$. Note that the regions GEJ and IBD have equal areas, as do ABC and FED.

Let $\mathcal{A}_1$ be the area of region GFH, and let $\mathcal{A}_2$ be the area of the region KFG. Then the area of $\mathcal{R}_1$ is $\mathcal{A}_1 + \mathcal{A}_2$.

Let $\mathcal{B}_1$ be the area of region GEJ (= area of region IBD), and let $\mathcal{B}_2$ be the area of the region FED (= area of region ABC).

We thus want to find b such that $\mathcal{B}_1 + (\mathcal{A}_2 - \mathcal{B}_2) = \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)$.

$\mathcal{A}_1 = \displaystyle\int_{\frac{\pi}{2}}^{\pi} \sin x\, dx = \int_0^{\frac{\pi}{2}} \cos x\, dx = 1$

$\mathcal{A}_2 = \displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx = \sqrt{2} - 1$

The coordinates of D are $(\cos^{-1} b, b)$, so that

$\mathcal{B}_1 = \displaystyle\int_0^{\cos^{-1} b} (\cos x - b)\, dx = \sin(\cos^{-1} b) - b\, \cos^{-1} b$

$\mathcal{B}_2 = \displaystyle\int_{\cos^{-1} b}^{\frac{\pi}{2}} (b - \cos x)\, dx = b\frac{\pi}{2} - b\cos^{-1} b - 1 + \sin(\cos^{-1} b)$

Thus, the condition $\mathcal{B}_1 + (\mathcal{A}_2 - \mathcal{B}_2) = \frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2)$ becomes

$$\sin(\cos^{-1} b) - b\, \cos^{-1} b + \sqrt{2} - 1 - \frac{b\pi}{2} + b\cos^{-1} b + 1 - \sin(\cos^{-1} b) = \frac{1}{2}(1 + \sqrt{2} - 1).$$

Several things cancel nicely, giving $b = \dfrac{\sqrt{2}}{\pi}$.

2. Find, with proof, the sum of the series

$$\sum_{n=1}^{\infty} \sin\frac{1}{2^{n+1}} \cos\frac{3}{2^{n+1}}.$$

Solution.

Using the identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$, the nth term of the series is $\dfrac{1}{2}\left[\sin\left(\dfrac{1}{2^{n-1}}\right) + \sin\left(\dfrac{-1}{2^n}\right)\right] = \dfrac{1}{2}\left[\sin\left(\dfrac{1}{2^{n-1}}\right) - \sin\left(\dfrac{1}{2^n}\right)\right]$ This yields a telescoping sum whose nth partial sum equals $\frac{1}{2}(\sin 1 - \sin\frac{1}{2^n})$. As n approaches infinity, the second term approaches 0, so the sum of the series is $\frac{1}{2}\sin 1$.

From a Michigan MATH Challenge.

3. Find, with proof, all positive integers x satisfying $3^{(2^{x!})} = 2^{(3^{x!})} + 1$.

Solution.

Clearly, $x = 1$ is a solution. If $x > 1$, then $x!$ is even and so $3^{x!} \equiv 1 \mod 4$. If $x > 1$ is a solution, then this implies that the last digit of $2^{(3^{x!})} + 1$ is 3. But this is impossible since the last digit of an even power of 3 is either 1 or 9. Thus, $x = 1$ is the unique solution.

This problem appeared in the 1985 Romanian Mathematical Olympiad.

4. Find a polynomial with integer coefficients that has $2^{\frac{1}{5}} + 2^{-\frac{1}{5}}$ as a root.

Solution.

Let $x = 2^{\frac{1}{5}} + 2^{-\frac{1}{5}}$ so that

$$x^3 = 2^{\frac{3}{5}} + 2^{-\frac{3}{5}} + 3(2^{\frac{1}{5}} + 2^{-\frac{1}{5}}) = 2^{\frac{3}{5}} + 2^{-\frac{3}{5}} + 3x,$$

or

$$2^{\frac{3}{5}} + 2^{-\frac{3}{5}} = x^3 - 3x.$$

Also,

$$\begin{aligned} x^5 &= 2 + 2^{-1} + 5(2^{\frac{3}{5}} + 2^{-\frac{3}{5}}) + 10(2^{\frac{1}{5}} + 2^{-\frac{1}{5}}) \\ &= 5/2 + 5(x^3 - 3x) + 10x. \end{aligned}$$

Hence, $2^{\frac{1}{5}} + 2^{-\frac{1}{5}}$ is a root of

$$P(x) = 2x^5 - 10x^3 + 10x - 5.$$

It is easy to adapt this argument to find a polynomial with integer coefficients that has $q^{\frac{1}{n}} + q^{-\frac{1}{n}}$ as a root, where q is a nonzero rational and n is an odd positive integer.

Problem 1187 from Crux Mathematicorum, Vol. 14, No. 1 (January 1988), pp. 30–31. Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire and solution by Bruce Shawyer, Memorial University of Newfoundland, St. John's.

5. For each positive integer n, let

$$\mathcal{R}_n = \left\{(x, y) : 0 \le x \le n \text{ and } 0 \le y \le \sqrt{x}\right\}.$$

Let $N(n)$ denote the number of points in $\mathcal{R}_n$ whose coordinates are both integers. Compute

$$\lim_{n\to\infty} \frac{N(n)}{n^{3/2}}.$$

Solution.

For each positive integer n, the number of integers y with $0 \le y \le \sqrt{n}$ is $\lfloor\sqrt{n}\rfloor + 1$, so that

$$N(n) = \sum_{k=0}^{n} \left(\lfloor\sqrt{k}\rfloor + 1\right) = (n+1) + \sum_{k=1}^{n} \lfloor\sqrt{k}\rfloor.$$

Since $\sqrt{k} - 1 < \lfloor \sqrt{k} \rfloor \le \sqrt{k}$, and

$$\int_0^n \sqrt{x}\,dx \le \sum_{k=1}^{n} \sqrt{k} \le \int_0^{n+1} \sqrt{x}\,dx,$$

we have

$$1 + \int_0^n \sqrt{x}\,dx \le N(n) \le (n+1) + \int_0^{n+1} \sqrt{x}\,dx,$$

and so

$$1 + \frac{2}{3}n^{3/2} \le N(n) \le (n+1) + \frac{2}{3}(n+1)^{3/2}.$$

Dividing by $n^{3/2}$ and applying the squeeze theorem gives

$$\lim_{n\to\infty} \frac{N(n)}{n^{3/2}} = \frac{2}{3}.$$

2016 Missouri Collegiate Mathematics Competition
Session II

1. Evaluate

$$\sum_{k=1}^{n} k2^{k-1}$$

for $n = 1, 2, 3, \ldots$.

Solution.

Let

$$f(x) = \sum_{k=0}^{n} x^k,$$

then

$$f'(x) = \sum_{k=1}^{n} kx^{k-1}$$

and

$$f'(2) = \sum_{k=1}^{n} k2^{k-1}.$$

Also,

$$f'(x) = \frac{d}{dx}\left(\frac{x^{n+1}-1}{x-1}\right) = \frac{(x-1)((n+1)x^n) - (x^{n+1}-1)(1)}{(x-1)^2}$$

and

$$f'(2) = 2^n(n+1) - 2^{n+1} + 1 = 2^n(n-1) + 1.$$

2. Evaluate

$$\int_1^{\infty} \frac{\lfloor x \rfloor}{x^3}\, dx,$$

where $\lfloor x \rfloor$ denotes the greatest integer function.

Solution.

Denoting the integral by I, we have

$$I = \sum_{k=1}^{\infty} \int_k^{k+1} \frac{\lfloor x \rfloor}{x^3}\, dx.$$

On the interval $[k, k+1)$, we make the substitution $x = k+\alpha$, with $0 \le \alpha < 1$. Then

$$\begin{aligned}
I &= \sum_{k=1}^{\infty} \int_{\alpha=0}^{\alpha=1} \frac{k}{(k+\alpha)^3}\, d\alpha \\
&= -\frac{1}{2}\sum_{k=1}^{\infty} \frac{k}{(k+\alpha)^2}\Bigg|_{\alpha=0}^{\alpha=1} \\
&= -\frac{1}{2}\sum_{k=1}^{\infty} \left[\frac{k}{(k+1)^2} - \frac{1}{k}\right] \\
&= -\frac{1}{2}\sum_{k=1}^{\infty} \left[\frac{k+1}{(k+1)^2} - \frac{1}{k} - \frac{1}{(k+1)^2}\right] \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{k^2} \\
&= \frac{1}{2} + \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) \\
&= \frac{\pi^2}{12}.
\end{aligned}$$

3. Find the smallest positive angle θ in degrees satisfying

$$\sin^2(2015\theta) + \cos^2(2016\theta) = 1.$$

Solution.

Using the identity $1 - \sin^2(2015\theta) = \cos^2(2015\theta)$, the given equation can be rewritten as $\cos^2(2016\theta) = \cos^2(2015\theta)$ which is equivalent to $\cos(2016\theta) = \pm\cos(2015\theta)$. As $\cos x$ is decreasing for $x \in [0, \pi]$, we have $\cos(2016\theta) \ne +\cos(2015\theta)$ for $\theta \in (0, \pi/2016]$. We find the $\theta \in (0, \pi/2016]$ for which $\cos(2016\theta) = -\cos(2015\theta)$. Equality occurs if and only if $2016\theta = \pi/2 + t$ and $2015\theta = \pi/2 - t$ for some positive t in the appropriate range. Solving for θ, we obtain $\theta = \pi/4031$. Hence, the answer in degrees is $(180/4031)^\circ$.

Similar to a Univ. of South Carolina practice problem for the American Regions Mathematics Competition.

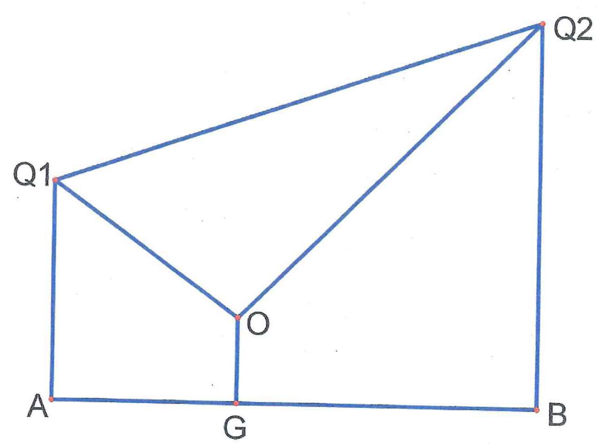
4. If a and b are the radii of two spheres, tangent to each other and to a plane

(on the same side of the plane), the radius x of the largest sphere which can pass between them is given by what formula?

Solution.

This is really a two-dimensional problem, so we replace the spheres by circles and the plane by a line l. Let $\mathcal{C}_1$ be the circle of radius $a, \mathcal{C}_2$ the circle of radius b, and $\mathcal{C}$ the circle of radius x. Let A be the point of tangency of $\mathcal{C}_1$ and l, B the point of tangency of $\mathcal{C}_2$ and l, and D the point of tangency of $\mathcal{C}_1$ and $\mathcal{C}_2$. Let Q_1 be the center of $\mathcal{C}_1$, Q_2 be the center of $\mathcal{C}_2$, and O the center of $\mathcal{C}$. Let E be the point of tangency of $\mathcal{C}_1$ and $\mathcal{C}$ and F the point of tangency of $\mathcal{C}_2$ and $\mathcal{C}$. Let G be the point of tangency of $\mathcal{C}$ and l.

Since D is on $\overline{Q_1Q_2}$, $Q_1Q_2 = a + b$. Since E and F are on $\overline{OQ_1}$ and $\overline{OQ_2}$, respectively, $OQ_1 = a + x$ and $OQ_2 = b + x$. Set $u = AG$ and $v = BG$.



We have the following system of equations.

$$\begin{cases} (a-x)^2 + u^2 = (a+x)^2 \\ (b-x)^2 + v^2 = (b+x)^2 \\ (b-a)^2 + (u+v)^2 = (a+b)^2 \end{cases}.$$

Expanding and simplifying gives

$$\begin{cases} u^2 = 4ax \\ v^2 = 4bx \\ (u+v)^2 = 4ab \end{cases}.$$

Substituting u and v from the first two of these into the third gives

$$\begin{aligned}\left(\sqrt{4ax}+\sqrt{4bx}\right)^2 &= 4ab\\ \sqrt{ax}+\sqrt{bx} &= \sqrt{ab}\\ \sqrt{x}\left(\sqrt{a}+\sqrt{b}\right) &= \sqrt{ab}\\ \sqrt{x} &= \frac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}}\\ x &= \frac{ab}{(\sqrt{a}+\sqrt{b})^2}.\end{aligned}$$

American Mathematical Monthly, E432, ODMPB p. 56.

5. In a regular octagon each side is colored blue or yellow. From such a coloring, another coloring will be obtained "in one step" as follows: if the two neighbors of a side have different colors, the "new" color of the side will be blue, otherwise the color will be yellow. (Note that the colors are modified simultaneously.) Show that after a finite number, say N, of moves, all sides will be colored yellow. What is the least value of N that works for all possible colorings?

Solution.

Consider a coloring of the sides of a regular octagon where each side is colored blue or yellow. Label the sides of the octagon S_1, …, S_8 and code the colors of the sides of the regular octagon by the column $\overline{x} = (x_1, \ldots, x_8)^T$, where

$$x_i = \begin{cases} 1 & \text{if } S_i \text{ is colored blue,} \\ 0 & \text{if } S_i \text{ is colored yellow,} \end{cases}$$

for $i = 1, 2, \ldots, 8$. Let

$$A = \begin{pmatrix} 0&1&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1\\ 1&0&0&0&0&0&1&0 \end{pmatrix}.$$

Then $A^n\overline{x}$ is the coded coloring of the sides of the regular octagon after n steps, where the addition and multiplication in the matrix operations are performed mod

2. Since $A^4 = 0$ it follows that after 4 steps, all sides will be colored yellow. This is the least possible value of N that works for all possible colorings, since for $\overline{x} = (1, 0, 0, 0, 0, 0, 0, 0)^T$,

$$A\overline{x} = (0, 1, 0, 0, 0, 0, 0, 1)^T, \quad A^2\overline{x} = (0, 0, 1, 0, 0, 0, 1, 0)^T, \quad A^3\overline{x} = (0, 1, 0, 1, 0, 1, 0, 1)^T.$$

Problem 7 from the 11th Austrian-Polish Mathematics Competition - December 1989.