2015 Missouri Collegiate Mathematics Competition
Session I

1. Find numbers a and b such that

$$\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x}=1.$$

Solution.

$$\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x}\cdot\frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2}=\lim_{x\to 0}\frac{ax+b-4}{x(\sqrt{ax+b}+2)}.$$

Since the denominator approaches 0 as x approaches 0, the numerator must also approach 0 for the limit to exist. So $a(0)+b-4=0$, and b must equal 4. Thus,

$$\lim_{x\to 0}\frac{a}{\sqrt{ax+4}+2}=1,$$

so

$$\frac{a}{\sqrt{4}+2}=1$$

and $a=4$.

2. If p, q, and r are distinct roots of $x^3-x^2+x-2=0$, find the value of $p^3+q^3+r^3$.

Solution.

We have

$$\begin{aligned}x^3-x^2+x-2&=(x-p)(x-q)(x-r)\\&=x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr,\end{aligned}$$

so $p+q+r=1$, $pq+pr+qr=1$, and $pqr=2$. We also know that p, q, and r are solutions of the equation, so

$$p^3+q^3+r^3-(p^2+q^2+r^2)+(p+q+r)-6=0.$$

Squaring $p+q+r=1$, we obtain $p^2+q^2+r^2+2(pq+pr+qr)=p^2+q^2+r^2+2=1$, therefore $p^2+q^2+r^2=-1$. We conclude that $p^3+q^3+r^3=-1-1+6=4$.

Problem 27 in the 1975 AHSME.

3. Show that for each positive integer n, the function $f_n(x)=x^n+(x-1)^n-(x+1)^n$

has a unique nonzero real root r_n and that $r_n \leq r_{n+1}$ for all n.

Solution.

First note that $f_n(0) = 0$ if n is even and $f_n(0) = -2$ if n is odd. Clearly $r_1 = 2$ is the unique real root of $f_1(x)$. Using $n = 1$ as the base case, the fact that $f'_{n+1}(x) = (n+1)f_n(x)$, and induction, we obtain the desired result.

This problem appeared in the 1995 issue of the Mathematics Magazine.

4. Consider the set S of all integer-valued triples (x, y, z) satisfying $6x+10y-15z = 1$. Find all such triples satisfying $0 \leq x, y, z \leq 2015$ and identify all such triples which give the maximum possible value of $y + z$.

Solution.

Modulo 3 the equation becomes $y = 1$, whence $y = 3s + 1$ for some integer s. Using this substitution and simplifying, the original equation becomes $2x - 5z = -10s - 3$. Modulo 2, this new equation becomes $z = 1$, whence $z = 2t + 1$ for some integer t. Therefore all integer solutions are of the form $(x, y, z) = (1-5s+5t, 3s+1, 2t+1)$ for some pair of integers s and t. With the additional restriction that $0 \leq x, y, z \leq 2015$, we require that $0 \leq s \leq 671$, $0 \leq t \leq 1007$, and $0 \leq t - s \leq 402$. Clearly $s = 671$ and $t = 1007$ give the triple $(1681, 2014, 2015)$ with the unique maximum value of $y + z = 4029$.

5. For each positive integer n let $g(n)$ be the number of digits greater than 4 in the decimal expansion of 2^n. Is it true that

$$\sum_{n=1}^{\infty} \frac{g(n)}{2^n} = \frac{2}{9}?$$

Prove or disprove.

Solution.

Yes.

Let n be any positive integer and $s(n)$ denote the (base 10) digital sum of n. Now it is fairly easy to show that for any positive integer n,

$$s(2^{n+1}) = 2s(2^n) - 9g(n) \tag{1}$$

and

$$s(n) \leq 9 \log n + 9, \tag{2}$$

where $\log n$ denotes the base 10 logarithm of n. Using (1), we have that for any positive integer m,

$$\sum_{n=1}^{m} \frac{g(n)}{2^n} = \sum_{n=1}^{m} \left(\frac{s(2^n)}{9 \cdot 2^{n-1}} - \frac{s(2^{n+1})}{9 \cdot 2^n} \right).$$

But this second series is telescoping and so

$$\sum_{n=1}^{m} \left(\frac{s(2^n)}{9 \cdot 2^{n-1}} - \frac{s(2^{n+1})}{9 \cdot 2^n} \right) = \frac{2}{9} - \frac{s(2^{m+1})}{9 \cdot 2^m}.$$

Finally, taking the limit as $m \to \infty$ and using (2) we have that

$$\sum_{n=1}^{\infty} \frac{g(n)}{2^n} = \frac{2}{9}.$$

Problem 6609 from the Advanced Problems in the American Mathematical Monthly, Vol. 96, No. 8 (October 1989), p. 743. Proposed by Doug Bowman and Tad White, University of California at Los Angeles.

2015 Missouri Collegiate Mathematics Competition
Session II

1. A smooth curve crosses the y-axis at the point $(0, 4)$ and is such that given any point P on the curve, the tangent line to the curve at P crosses the x-axis at a point Q whose x-coordinate is 2015 more than the x-coordinate of P. Determine the area of the region in the first quadrant bounded by the x-axis, the y-axis, and this curve.

Solution.

The slope of the tangent line to the curve at P is

$$\frac{dy}{dx} = \frac{-y}{2015},$$

so

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{2015},$$

and it follows that

$$\frac{d}{dx}\ln y = \frac{-1}{2015} \quad \text{and} \quad \ln y = \frac{-x}{2015} + C.$$

Thus,

$$y = Ce^{-x/2015}$$

and since the curve passes through $(0, 4)$,

$$y = 4e^{-x/2015}.$$

The area we are after is

$$\int_0^\infty y\,dx = \lim_{t\to\infty}\int_0^t 4e^{-x/2015}\,dx = \lim_{t\to\infty} -8060e^{-t/2015} + 8060 = 8060.$$

2. Consider the following two-person game: Start with 2015 pennies on a table. Players A and B alternate turns, with A going first. A legal move consists of removing any divisor of the number of pennies on the table, as long as the divisor is *strictly less than* the number of pennies on the table. For example, at the start of the game, player A can remove 1, 5, 13, 31, 65, 155, or 403 pennies, but not 2015 pennies. If player A removes 5 pennies, then player B could remove, say, 3 pennies, etc.

The game ends when no legal move is possible (i.e., when only one penny remains), and whoever's turn it is loses (and the opponent wins). So the objective is to leave your opponent with just one penny. Is there a winning strategy? If so, who wins, A or B?

Solution.

Any position with an even number of pennies is a winning position and any position with an odd number of pennies is a losing position. This can be shown using strong induction. First notice that 2 pennies is a winning position and 1 and 3 are losing positions (with 3, the player has no choice but to take 1, forcing the opponent to win by taking 1). Now suppose that any even number less than $2n$ is a winning position and any odd number less than $2n$ is a losing position. Then a player with $2n$ pennies can win by taking 1 penny (leaving an odd number less than $2n$, a losing position). However, a player with $2n + 1$ pennies cannot win. He or she is forced to take an odd number of pennies (since all divisors of an odd number are odd), leaving his or her opponent with an even number of pennies less than $2n$, a winning position.

So Player A can never win as long as B adopts a winning strategy such as taking 1 penny at every turn.

Similar to a Michigan MATH Challenge problem.

3. A certain game of chance involves choosing 20 distinct integers from the set $\{1, 2, \ldots, 80\}$. While waiting to see if this selection would hypothetically lead to winning a huge sum of money, you wonder if there always exist two nonempty and disjoint subsets of the chosen 20 integers having equal sums of squares of their elements. Show that, yes, two such subsets always exist.

Solution.

Suppose that the 20 numbers are $\{a_1, a_2, \ldots, a_{20}\}$. The number of distinct subsets of this set is $2^{20} > 10^6$. Also note that $0 < a_1^2 + a_2^2 + \cdots + a_{20}^2 < 20 \cdot 80^2 = 128000 < 10^6$. By the Pigeonhole Principle, we can see that there must be two distinct subsets with the same sum of squares. if the two subsets are not disjoint, remove the elements in common to leave two disjoint subsets with the same sum of squares. Since at least one of the sets is nonempty, the sum is nonzero, which means the other set also must be nonempty.

Similar to a Michigan MATH Challenge problem.

4. Prove that the sum of any finite subsequence of consecutive terms of the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ written in lowest terms has an odd numerator.

Solution.

Suppose that 2^m is the highest power of 2 dividing any of the denominators of the terms of the given finite subsequence (all written in lowest terms). If 2^m divided two such denominators, say $x \cdot 2^m$ and $y \cdot 2^m$ with x and y necessarily odd, then there must be some even integer $z \in (x, y)$. But then $z \cdot 2^m$ is also a denominator in one of the terms of the finite subsequence. But then $2^{m+1} \mid z \cdot 2^m$ contradicts the maximality of m. Therefore, the least common denominator of the terms in this finite sequence of rational numbers is $w \cdot 2^m$ for some odd integer w. Now, with each term written with this common denominator, all but one term (the unique term with with 2^m dividing its denominator after it has been written in lowest terms) has an even numerator. Therefore the sum, after it is written in lowest terms, must have an odd numerator.

From the May 1998 issue of the College Mathematics Journal.

5. Let $x_1 = 1$ and

$$x_{n+1} = \frac{1}{x_n}\left(\sqrt{1+x_n^2} - 1\right)$$

for $n \geq 1$. Show that the sequence $\{2^n x_n\}$ converges and finds its limit.

Solution.

We first show by induction on n that

$$x_n = \tan(\pi/2^{n+1}) \tag{3}$$

for all positive integers n. Since $x_1 = 1 = \tan(\pi/4)$, (3) is true for $n = 1$. Suppose (3) is true for some positive integer n. Then

$$\begin{aligned} x_{n+1} &= \frac{1}{\tan(\pi/2^{n+1})}\left(\sqrt{1+\tan^2(\pi/2^{n+1})} - 1\right) \\ &= \frac{\sec(\pi/2^{n+1}) - 1}{\tan(\pi/2^{n+1})} = \frac{1-\cos(\pi/2^{n+1})}{\sin(\pi/2^{n+1})} \\ &= \tan\left(\frac{1}{2}\cdot \pi/2^{n+1}\right) = \tan(\pi/2^{n+2}). \end{aligned}$$

So, by induction, (3) is true for all positive integers n.

Let $y = 1/2^n$. Then as $n \to \infty$, $y \to 0^+$, so by L'Hôpital's rule

$$\lim_{n\to\infty} 2^n x_n = \lim_{y\to 0^+} \frac{\tan(\pi y/2)}{y} = \lim_{y\to 0^+} \frac{\pi}{2}\cdot\frac{\sec^2(\pi y/2)}{1} = \frac{\pi}{2}.$$

Problem 1505 from Crux Mathematicorum, Vol. 17, No. 3 (March 1991), p. 86. Proposed by Marcin E. Kuczma and solution by Jeff Higham.