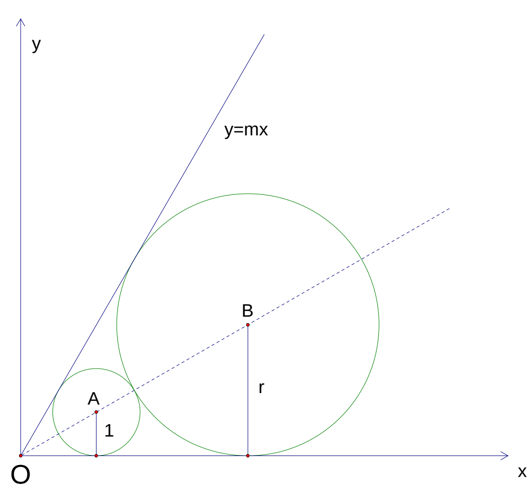
2012 Missouri Collegiate Mathematics Competition
Session I

1. Two circles in the first quadrant are tangent to each other and both are tangent to the x-axis and the line $y = mx$, where m is a positive constant. Find the ratio of the radius of the larger circle to the radius of the smaller circle as a function of θ, the angle between the x-axis and the line $y = mx$.

Solution.

Solution (see attached diagram): Let O be the origin, A the center of the smaller circle, and B the center of the larger circle. The bisector of angle O goes through A and B. Without loss of generality, let the radius of the smaller circle be equal to 1, and let r be the radius of the larger circle, x the length of segment $\overline{OA}$, and y the length of segment $\overline{OB}$.

By similar triangles, we have $x/1 = y/r$, so $y = rx$. By considering segment $\overline{OB}$, we also find that $x+1+r = y$. Therefore, we have $x+1+r = rx$ or $r = (x+1)/(x-1)$. Since $y = \csc(\theta/2)$, we find that $r = \dfrac{\csc(\theta/2)+1}{\csc(\theta/2)-1}$, or equivalently $r = \dfrac{1+\sin(\theta/2)}{1-\sin(\theta/2)}$.



(Similar to a C of C problem.)

2. Prove that $AB - BA \neq I_n$ for any $n \times n$ matrices A and B over the real numbers, where I_n denotes the $n \times n$ identity matrix.

Solution.

We consider the sum of the elements of main diagonals. Let $A = (a_{ij})$ and $B = (b_{ij})$. Then the sum of the elements of the main diagonal of $AB - BA$ is $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} - \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji} = 0$ while the sum of the main diagonal of I_n is n. Therefore, $AB-BA \neq I_n$.

3. Find

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n}}\right).$$

Solution.

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n}}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \cdots + \sqrt{\frac{n}{n+n}}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+2/n}} + \cdots + \frac{1}{\sqrt{1+1}}\right)\\
&= \int_0^1 \frac{1}{\sqrt{1+x}}\,dx = 2\sqrt{1+x}\Big|_0^1 = 2(\sqrt{2}-1).
\end{aligned}$$

4.

Part 1. Let n be a positive integer and let $P_{n-1}(x)$ be the polynomial equal to $P_{n-1}(x) = \frac{x^n-1}{x-1}$. Find $P_{n-1}(1)$.

Part 2. On the circle of radius 1, let V_1, V_2, …, V_n be the vertices of a regular n-gon inscribed in the circle. Let $\lambda_k = dist(V_1, V_k)$, $k = 1, 2, \ldots, n$. Show that

$$\prod_{k=2}^{n} \lambda_k = n.$$

Part 3. For the regular n-gon in Part 2, find the product of the lengths of all the line segments joining the vertices.

Solution.

Part 1. $P_{n-1}(x) = n$, either by using L'Hopital's Rule to evaluate $\lim_{x\to 1} P_{n-1}(x)$, or by factoring the numerator as $(x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$ to see that $P_{n-1}(x)$ is the second factor.

Part 2. Consider the vertices as the nth roots of 1 and let $V_1 = 1$. Then the V_k's are complex numbers and

$$\prod_{k=1}^{n}(z - V_k) = z^n - 1$$

because the product is simply the factored form of $z^n - 1$. Thus,

$$\prod_{k=2}^{n}(z - V_k) = P_{n-1}(z)$$

and $\lambda_k = |V_1 - V_k| = |1 - V_k|$. Therefore, from Part 1,

$$\prod_{k=2}^{n}\lambda_k = \prod_{k=2}^{n}|1 - V_k| = |P_{n-1}(1)| = n.$$

Part 3. From Part 2, for each vertex, the product of the lengths of the segments to the other vertices is n, so the product for all vertices is n^n. But because each segment joins two vertices, every segment's length appears twice as a factor in n^n. Thus, if the required product is P, we have $P^2 = n^n$ giving $P = \sqrt{n^n}$.

Parts 1 and 2 are from de Souza, P. N. and Silva, J-N, *Berkeley Problems in Mathematics*, Springer, 1998.

5. Let S be a set of twelve distinct positive integers such that for distinct a, b, c, and d in S, $a + b \neq c + d$. Prove that the largest element in S is greater than 56.

Solution.

Consider the $\binom{12}{2} = 66$ possible differences $b - a$, where $b > a$ are elements of S. We observe that if $a, b, c, d \in S$ are all distinct and $a - c = d - b$, then $a + b = c + d$. And if $a, b, c \in S$ are all distinct and $c - a = b - c$, then $c = (a + b)/2$ so c is not the largest or smallest element of S. And if $c - a_1 = b_1 - c$ and $c - a_2 = b_2 - c$, then

$a_1 + b_1 = 2c = a_2 + b_2$. So for each c, not the largest or smallest element of S, at most one pair of a, b can be found such that $c - a = b - c$. Hence, we can have at most 10 duplicates for each c not the largest or smallest element of S. Therefore, there are at least $66 - 10 = 56$ distinct differences. Hence, the largest element of S is greater than 56.

In fact, one such set of 12 numbers is

$$\{1, 2, 3, 8, 13, 23, 38, 41, 55, 64, 68, 72\}.$$

This problem is from G. Berzsenyi, *Math Investigations: Distinct Sums of Twosomes*, Quantum, **5.4** (March/April 1995), 39.

2012 Missouri Collegiate Mathematics Competition
Session II

1. Express 2012 as a sum of (two or more) consecutive integers.

Solution.

We need to find positive integers k and m, $m \geq 2$, such that

$$(k+1)+(k+2)+\cdots+(k+m)=2012.$$

Note that the left side of this equation is the difference between the sum of the first $k+m$ positive integers and the sum of the first k positive integers, so we have

$$\begin{aligned}2012 &= \frac{(k+m)(k+m+1)}{2}-\frac{k(k+1)}{2}\\ &= \frac{1}{2}(k^2+2km+m^2+k+1-k^2-k)\\ &= \frac{1}{2}m(m+2k+1).\end{aligned}$$

Thus,

$$m(m+2k+1)=4024=8\cdot 503.$$

Note that m and $m+2k+1$ have opposite parity and $m+2k+1>m>1$. Therefore, m must be 8 and k must be 247. The desired sum is then $2012 = 248+249+250+251+252+253+254+255$.

(Similar to an Iowa contest problem by Jacek Fabrykowski.)

2. Let f be a function that has a continuous derivative over the interval $[a,b]$, and let $f(a)=f(b)=0$. Prove that

$$\max_{a\leq x\leq b}|f'(x)| \geq \frac{4}{(b-a)^2}\int_a^b |f(x)|dx.$$

Solution.

By the mean value theorem, for any $x \in (a,b)$,

$$f(x)=f(x)-f(a)=f'(\xi_1)(x-a)=f(x)-f(b)=f'(\xi_2)(x-b),$$

where $\xi_1 \in (a,x)$ and $\xi_2 \in (x,b)$. Let $M=\max\limits_{a\leq x\leq b}|f'(x)|$. Then, $|f(x)| \leq M(x-a)$

and $|f(x)| \le M(b-x)$. Therefore,

$$\begin{aligned} & \frac{4}{(b-a)^2}\int_a^b |f(x)|dx \\ \le\ & \frac{4}{(b-a)^2}\left(\int_a^{(a+b)/2} M(x-a)dx + \int_{(a+b)/2}^b M(b-x)dx\right) \\ =\ & \frac{4}{(b-a)^2}\left(\frac{(b-a)^2}{8}M + \frac{(b-a)^2}{8}M\right) = M. \end{aligned}$$

3. For real $a > 0$ define the sequence $\{x_n\}$ by

$$x_{n+1} = a(x_n^2 + 4), \qquad x_0 = 0.$$

Determine necessary and sufficient conditions on a for $\lim_{n\to\infty} x_n$ to exist and be finite.

Solution.

Assume $\lim_{n\to\infty} x_n = x$, with x finite. Then $x = a(x^2+4)$, making

$$x = \frac{1}{2a}\left(1 \pm \sqrt{1-16a^2}\right)$$

To make x real and finite, then, we need $1 - 16a^2 \ge 0$, and since $a > 0$ was given, we get $0 < a \le \frac{1}{4}$.
Now assume $0 < a \le \frac{1}{4}$. By induction, we can show that $\{x_n\}$ is a nondecreasing sequence.

$$x_1 = 4a > x_0 = 0.$$

If we assume $x_n \ge x_{n-1}$, then

$$x_{n+1} - x_n = a(x_n^2 - x_{n-1}^2)$$

and so $x_{n+1} \ge x_n$.
Note that $x_1 = 4a < 2$, so another induction will show that $\{x_n\}$ is bounded above. Let $x_n < 2$. Then

$$x_{n+1} = a(x_n^2 + 4) < \frac{1}{4}(4+4) = 2.$$

This means $\{x_n\}$ is a nondecreasing sequence that is bounded above, and thus $\{x_n\}$ converges to a real finite limit.
The required necessary and sufficient condition is $0 < a \le \frac{1}{4}$.

4. Let $P(x)$ be a polynomial in x of degree $n > 2$. Suppose that $a \ne b$, $P(a) = n_1$, and $P(b) = n_2$. What is the remainder $R(x)$ when $P(x)$ is divided by $(x-a)(x-b)$?

Solution.

By the Factor Theorem for polynomials over a field,

$$\frac{P(x)}{x-a} = Q_1(x) + \frac{n_1}{x-a}, \quad \frac{P(x)}{x-b} = Q_2(x) + \frac{n_2}{x-b}, \tag{1}$$

where $Q_1(x)$, $Q_2(x)$ are unique polynomials of degree $n-1$. The Division Algorithm for polynomials over a field leads to the conclusion that if $P(x)$ is divided by $(x-a)(x-b)$, then there is a unique quotient $Q(x)$ and a unique remainder $cx+d$, $c, d \in \mathbb{R}$ such that

$$\frac{P(x)}{(x-a)(x-b)} = Q(x) + \frac{cx+d}{(x-a)(x-b)}. \tag{2}$$

Multiplications of eq. (2) by either $(x-b)$ or $(x-a)$ then lead to the equations

$$\frac{P(x)}{x-a} = (x-b)Q(x) + \frac{cx+d}{x-a} = ((x-b)Q(x)+c) + \frac{d+ac}{x-a},$$
$$\frac{P(x)}{x-b} = (x-a)Q(x) + \frac{cx+d}{x-b} = ((x-a)Q(x)+c) + \frac{d+bc}{x-b}.$$

Comparison of these results with the equations in (1) then imply (since $a \neq b$) if $d+ac = n_1$, $d+bc = n_2$. Thus, $c = \frac{n_2-n_1}{b-a}$ and $d = \frac{n_1 b - n_2 a}{b-a}$, and the desired remainder is

$$R(x) = \frac{1}{b-a}\left((n_2-n_1)x + (n_1 b - n_2 a)\right).$$

5. Let S be the set of points in $\mathbb{R}^2$ that constitute the graph of $y = x^2$, $-2 \leq x \leq 2$, and let $d(p_1, p_2)$ denote the Euclidean distance between $p_1, p_2 \in S$. Determine p_1 and p_2 that maximizes $d(p_1, p_2)$.

Solution.

It is geometrically apparent that one of the points of a maximally separated pair $\{p_1, p_2\}$ must be either $(2,4)$ or $(-2,4)$. Consider the point $(-2,4)$ as p_2 and (x, x^2) as p_1. Let $g(x) = d^2(p_1,p_2) = (x+2)^2 + (x^2-4)^2$. Then

$$g'(x) = 2(x+2) + 2(x^2-4)\cdot 2x = 2(x+2)(2x^2-4x+1).$$

Setting this equal to 0 and eliminating trivial conditions we have

$$x^{\pm} = 1 \pm \frac{\sqrt{2}}{2}.$$

Calculating the second derivative we have

$$g''(x) = 2\left(2x^2 - 4x + 1 + (x+2)(4x-4)\right) = 2(6x^2-7).$$

Using the second derivative test, we find that $g''(x^-) < 0$ and $g''(x^+) > 0$, so the distance is maximized at x^-. Therefore, the points which maximize the distance are

$$(-2, 4) \quad \text{and} \quad \left(1 - \frac{\sqrt{2}}{2}, \left(1 - \frac{\sqrt{2}}{2}\right)^2\right)$$

and by symmetry

$$(2, 4) \quad \text{and} \quad \left(-1 + \frac{\sqrt{2}}{2}, \left(-1 + \frac{\sqrt{2}}{2}\right)^2\right).$$