2011 Missouri Collegiate Mathematics Competition
Session I

1. Consider the parabola $x^2 = 4\,p\,y$, with vertex at the origin, focus at $F(0, p)$, and directrix $y = -p$. Let $P(x, y)$ be a point on the parabola and let $R(x, -p)$ be the intersection of the vertical line through P with the directrix. Determine P so that the triangle PFR is equilateral.

Solution.

By the definition of a parabola,

$$dist(F, P) = dist(P, R) = \frac{x^2 + 4p^2}{4p}.$$

$dist(F, R) = \sqrt{4p^2 + x^2}$, so we need

$$\frac{x^2 + 4p^2}{4p} = \sqrt{4p^2 + x^2}$$

which is equivalent to

$$\sqrt{4p^2 + x^2} = 4p$$

making $x = 2\sqrt{3}p$ and $y = 3p$.

2. Prove that the inequality $\sin x + \arcsin x > 2x$ holds for all values of x such that $0 < x \le 1$.

Solution I.

The Maclaurin series for $\sin x$ and $\arcsin x$ are given by

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \arcsin x &= x + \left(\frac{1}{2}\right) \cdot \frac{x^3}{3} + \left(\frac{1}{2} \cdot \frac{3}{4}\right) \cdot \frac{x^5}{5} + \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}\right) \cdot \frac{x^7}{7} + \cdots . \end{aligned}$$

Therefore, for each $0 < x \le 1$, there exists a θ_1 and θ_2 between 0 and x such that

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{\theta_1^5}{5!} \\ \arcsin x &= x + \left(\frac{1}{2}\right) \cdot \frac{x^3}{3} + \left(\frac{1}{2} \cdot \frac{3}{4}\right) \cdot \frac{\theta_2^5}{5}. \end{aligned}$$

Therefore,

$$\sin x + \arcsin x = 2x + \frac{\theta_1^5}{5!} + \left(\frac{1}{2}\cdot\frac{3}{4}\right)\cdot\frac{\theta_2^5}{5} > 2x.$$

Solution II.

Let $f(x) = \sin x + \arcsin x$ and note that $f(0) = 0$. Then $f'(x) = \cos x + \dfrac{1}{\sqrt{1-x^2}}$ and $f'(0) = 2$, making $y = 2x$ the tangent line to the graph of f at the origin. Because $f''(x) = -\sin x + \dfrac{x}{(1-x^2)^{3/2}}$, and $\dfrac{x}{(1-x^2)^{3/2}} > x > \sin x$ for $0 < x < 1$, the graph of f is concave up on $(0,1)$, and thus above the tangent line at $(0,0)$, giving the required inequality.

This problem is from *How do you Figure? Math Challenges, M283*, Quantum, **10.3** (Jan/Feb 2000), 21, 46–47.

3. For what non-negative integers n does there exist a polynomial P_n of degree n with integer coefficients satisfying $P_n(k) = 2^k$ for all integers k, $0 \le k \le n$? Find all such polynomials.

Solution.

Suppose that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. First assume that $n > 1$. Since $P(0) = 1$, $a_0 = 1$. Then, since $P(2) = 4$, we have $2^n a_n + 2^{n-1}a_{n-1} + \cdots + 2a_1 = 3$. Since the parity of the left-hand-side and right-hand-side are different, no such polynomial exists. Now assume that $n \le 1$. When $n = 0$, the polynomial $P_0(x) = 1$ does the trick. When $n = 1$, the polynomial $P_1(x) = x + 1$ satisfies both $P_1(0) = 1$ and $P_1(1) = 2$. Moreover, since $P_n(0) = 1$ when $n = 0, 1$ and $P_1(1) = 2$, P_0 and P_1 are the only such polynomials.

4. Let a and b be real numbers with $b \ne 0$ and let $C(\mathbb{R})$ denote the set of continuous functions from the reals to the reals. Define $T : C(\mathbb{R}) \to C(\mathbb{R})$ by $(Tf)(x) = ax + b\int_0^x f(t)\,dt$. Find all fixed points of T^2, where T^2 denotes the composition of T with itself.

Solution.

A function f is a fixed point of T^2 if and only if for all real x

$$\begin{aligned} f(x) &= \left(T^2\right)(f)(x), \\ f(x) &= ax + b\int_0^x (Tf)(t)\,dt, \\ f(x) &= ax + b\int_0^x \left\{at + b\int_0^t f(s)\,ds\right\} dt. \end{aligned}$$

Since f is continuous, the right-hand side is a differentiable function of x; thus the left-hand side is also. Differentiating gives

$$f'(x) = a + abx + b^2\int_0^x f(s)\,ds.$$

Again, both sides are differentiable. Differentiating gives

$$f''(x) = ab + b^2 f(x).$$

This is a constant coefficient differential equation; the general form of the solution is $f(x) = c_1 e^{bx} + c_2 e^{-bx} + k$, for some constants c_1, c_2, and k. Substituting in the differential equation gives $k = -\dfrac{a}{b}$. From the above, the initial conditions are $f(0) = 0$ and $f'(0) = a$. Since $f'(x) = bc_1 e^{bx} - bc_2 e^{-bx}$, we have

$$\begin{cases} c_1 + c_2 - \dfrac{a}{b} = 0; \\ b(c_1 - c_2) = a, \end{cases}$$

so that $c_1 = \dfrac{a}{b}$ and $c_2 = 0$. Thus, the unique fixed point is

$$f(x) = \frac{a}{b}\left(e^{bx} - 1\right).$$

5. Let the function F be given by

$$F(x) = e^{-x} - \left(1 - \frac{x}{n}\right)^n.$$

Show that, for $n \geq 2$,

$$0 \leq F(x) \leq \frac{e^{-1}}{n} \qquad \text{on} \qquad [0, n].$$

Solution.

The function F is continuous and differentiable on $[0, n]$, and so has both a maximum and a minimum there, either at an end point or at a critical number. We have $F(0) = 0$, $F(n) = e^{-n}$, and

$$F'(x) = -e^{-x} + \left(1 - \frac{x}{n}\right)^{n-1}.$$

To show $F(x) \geq 0$, we need to show

$$\left(1 - \frac{x}{n}\right)^{n} \leq e^{-x} \qquad \text{on} \qquad [0, n],$$

or

$$\left(1 - \frac{x}{n}\right)^{n} \leq \left(e^{-\frac{x}{n}}\right)^{n} \qquad \text{on} \qquad [0, n],$$

or

$$0 \leq \left(1 - \frac{x}{n}\right) \leq e^{-\frac{x}{n}} \qquad \text{on} \qquad [0, n].$$

This is equivalent to showing

$$0 \leq (1 - t) \leq e^{-t} \qquad \text{on} \qquad [0, 1],$$

which is clear, because $y = 1 - t$ is the tangent line to $y = e^{-t}$ at the point $(0, 1)$, and the graph of $y = e^{-t}$ is concave up everywhere and thus is above all its tangent lines. So the minimum is 0, which is attained at $x = 0$.

Because $F'(n) = -e^{-n} < 0$, the maximum does not occur at $x = n$ and so must be at a point x_0 in $(0, n)$ where $F'(x_0) = 0$. This means

$$e^{-x_0} = \left(1 - \frac{x_0}{n}\right)^{n-1}$$

and

$$\begin{aligned} F(x_0) &= e^{-x_0} - \left(1 - \frac{x_0}{n}\right)^{n} \\ &= e^{-x_0} - \left(1 - \frac{x_0}{n}\right)^{n-1}\left(1 - \frac{x_0}{n}\right) \\ &= e^{-x_0} - e^{-x_0}\left(1 - \frac{x_0}{n}\right) = \frac{x_0 e^{-x_0}}{n}. \end{aligned}$$

Now let $g(x) = x\, e^{-x}$. Basic calculus shows that g has a maximum of e^{-1} at $x = 1$, and so

$$F(x_0) \leq \frac{e^{-1}}{n}.$$

This problem is a lemma in the article by Levrie and Daems, Evaluating the probability integral using Wallis' product formula for π, *The American Mathematical Monthly* 116, (2009) 538–541. Later in the paper, x is replaced by x^2 and $n \to \infty$.

2011 Missouri Collegiate Mathematics Competition
Session II

1. For $0 < r < 1$, let n_r denote the line that is normal to the curve $y = x^r$ at the point $(1, 1)$, and let S_r denote the region in the first quadrant of the xy-plane bounded by the x-axis, the curve $y = x^r$, and the line n_r. Find the value of r that minimizes the area of S_r.

Solution.

From $y' = rx^{r-1}$, the slope of the curve $y = x^r$ at $(1, 1)$ is r; hence the equation of line n_r is

$$y - 1 = -\frac{1}{r}(x - 1),$$

or equivalently,

$$x = r(1 - y) + 1.$$

The area of S_r is given by

$$\begin{aligned} A(r) &= \int_0^1 \left[r(1-y) + 1 - y^{1/r}\right] dy \\ &= \left(ry - \frac{1}{2}ry^2 + y - \frac{1}{\frac{1}{r}+1}y^{\frac{1}{r}+1}\right)\Bigg|_0^1 \\ &= r - \frac{1}{2}r + 1 - \frac{r}{r+1} \\ &= \frac{1}{2}r + \frac{1}{r+1}. \end{aligned}$$

The only critical value of $A(r)$ for $0 < r < 1$ is the unique solution in that interval to $\frac{1}{2} - \frac{1}{(r+1)^2} = 0$, namely, $r = \sqrt{2} - 1$. Since $A''(r) = \frac{2}{(r+1)^3}$, $A''\left(\sqrt{2}-1\right) > 0$, implying that the unique local minimum of $A(r)$ is at $r = \sqrt{2} - 1$. We have $A\left(\sqrt{2}-1\right) = \sqrt{2} - \frac{1}{2}$. Since $\lim_{r\to 0^+} A(r) = 1$ and $\lim_{r \to 1^-} A(r) = +\infty$, $A(r)$ has a global minimum at $r = \sqrt{2} - 1$.

2. A rectangular strip is divided into n bands of equal width as in the following figure.

1	2						n

Each band is colored by one of m colors. Two patterns are considered identical if one is a (left-to-right) mirror reflection of the other. Determine the number of distinct patterns with m colors and n bands.

Solution.

If a color assignment is not altered by a mirror reflection, then for any i, bands i and $n+1-i$ must have the same color. If n is even, we can assign arbitrary colors to bands $1, 2, \ldots, n/2$, in which case there are $m^{n/2}$ color schemes invariant under a mirror reflection. If n is odd, then the bands $1, 2, \ldots, (n+1)/2$ can be given an arbitrary coloring scheme and hence there are $m^{(n+1)/2}$ color schemes invariant under mirror reflection.

Since there are a total of m^n coloring schemes, and since we can put in pairs, the schemes which are not invariant under mirror reflection, there are a total of $m^{n/2}+\frac{1}{2}(m^n-m^{n/2}) = \frac{1}{2}(m^n+m^{n/2})$ distinct patterns when n is even and $m^{(n+1)/2}+\frac{1}{2}(m^n-m^{(n+1)/2}) = \frac{1}{2}(m^n+m^{(n+1)/2})$ distinct patterns when n is odd.

3. Let M be an arbitrary 3×3 matrix, each of whose entries m_{ij} is drawn from the set $\{-1, 1\}$, and let $\det M$ be the determinant of M. Prove that $|\det M| \leq 4$.

Solution I.

A 2×2 matrix with entries ± 1 necessarily has determinant 0 or ± 2. Expansion of the 3×3 $\det M$ in minors of the elements in the first row gives

$$\det M = m_{11}\begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12}\begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13}\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{33} \end{vmatrix}, \tag{1}$$

so the left-hand side must be even. Also, it is apparent from (1) that $|\det M|$ cannot exceed 6. It can be 6 only if none of the minors is 0. But in the 2×3 matrix

$$\begin{pmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix},$$

where the entries are 1's and -1's, it is always possible to find a 2×2 submatrix (by striking out some particular column) whose determinant is 0, regardless of the distribution of 1's and -1's. Thus, at least one of the terms in (1) must vanish, and we have $|\det M| \leq 4$.

Solution II.

Hadamard's Determinantal Inequality says that for any $n\times n$ real matrix M_n, $n > 2$, one has

$$|\det M_n| \leq \prod_{j=1}^{n}\left\{\sum_{i=1}^{n} m_{ij}^2\right\}^{1/2}.$$

For matrices of the sort considered here, the right-hand side of the inequality is $3\sqrt{3} \approx 5.2$. Again, since $\det M_3$ must be even, then $|\det M_3| \leq 4$ follows.

4. Calculate

$$\int_0^2 \left(\sqrt{1+x^3} + \sqrt[3]{x^2+2x}\right)\, dx.$$

Solution.

Consider the rectangle with the vertices $O(0,0)$, $A(2,0)$, $B(2,3)$, and $C(0,3)$ on the coordinate plane. The graph of the function $y=\sqrt{1+x^3}$ passes through the points $(0,1)$ and $(2,3)$ and partitions our rectangle into two parts. The area under the graph is

$$\int_0^2 \sqrt{1+x^3}\,dx.$$

We compute the area of the part of the rectangle that is above the graph. The function $y=\sqrt{1+x^3}$ is monotonic on the segment $[0,2]$. Thus, we can express x in terms of y:

$$x=\sqrt[3]{y^2-1}.$$

Therefore, the area we seek is

$$\int_1^3 \sqrt[3]{y^2-1}\,dy.$$

Make the following change of variable under the integral sign: $y=t+1$. This yields the following expression for the area:

$$\int_0^2 \sqrt[3]{t^2+2t}\,dt.$$

Now we see that the given integral is equal to the area of the rectangle $OABC$, which is 6.

This problem is from *How do you Figure? Math Challenges, M249*, Quantum, **9.2** (Nov/Dec 1998), 25, 45–46.

5. Evaluate the series

$$\sum_{n=0}^{\infty}\frac{1}{2011^{2^n}-2011^{-2^n}}=\frac{1}{2011^1-2011^{-1}}+\frac{1}{2011^2-2011^{-2}}+\frac{1}{2011^4-2011^{-4}}+\dots$$

and express it as a rational number.

Solution.

More generally, let

$$S(x)=\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+1}}},\quad S_N(x)=\sum_{n=0}^{N}\frac{x^{2^n}}{1-x^{2^{n+1}}}.$$

Note that the series in the problem is equal to $S(1/2011)$. We will show that, for any x with $0 < x < 1$,

$$S(x) = \frac{x}{1-x}. \tag{2}$$

We use a "telescoping" argument based on the elementary identity

$$\frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{1}{1-x^{2^n}} - \frac{1}{1-x^{2^{n+1}}}.$$

If we substitute this into the partial sum $S_N(x)$, we get

$$S_N(x) = \left(\frac{1}{1-x} - \frac{1}{1-x^2}\right) + \left(\frac{1}{1-x^2} - \frac{1}{1-x^4}\right) + \cdots + \left(\frac{1}{1-x^{2^N}} - \frac{1}{1-x^{2^{N+1}}}\right)$$

which equals $\dfrac{1}{1-x} - \dfrac{1}{1-x^{2^{N+1}}}$. As $N \to \infty$, the second term approaches 1, thus

$$S(x) = \lim_{N\to\infty} S_N(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x},$$

proving (2).
Therefore, the solution of the problem is $S(1/2011) = (1/2011)/(1 - 1/2011) = 1/2010$.

(Similar to a U of I contest problem.)