2010 Missouri Collegiate Mathematics Competition

Session I

1. For the parabola having equation $y = -x^2$ let $a < 0$ and $b > 0$ with points $P : (a, -a^2)$ and $Q : (b, -b^2)$. Let M be the midpoint of PQ and let R be the intersection of the vertical line through M with the parabola. Show that the area of the region bounded by the parabola and the line segment PQ is $\frac{4}{3}$ of the area of triangle PQR. (Archimedes, 3rd century B.C.)

Solution.

The equation of the line containing P and Q is

$$y = -(b + a)x + ab.$$

Thus, the area of the segment of the parabola is

$$\text{Area}(\text{seg}PQR) = \int_a^b (-x^2 + (b - a)x - ab)dx = \frac{(b - a)^2}{6}.$$

Triangle PQR is divided into two triangles by line segment MR and

$$|MR| = -\left(\frac{a + b}{2}\right)^2 + \frac{a^2 + b^2}{2} = \left(\frac{b - a}{2}\right)^2.$$

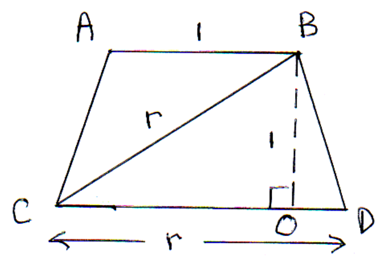
For each of these triangles, the altitude to MR is $(b - a)/2$, so they have equal areas, giving

$$\text{Area}(\triangle PQR) = \left(\frac{b - a}{2}\right)\left(\frac{b - a}{2}\right)^2 = \frac{(b - a)^3}{8}.$$

2. Suppose that the length of the larger base of an isosceles trapezoid equals the length of a diagonal and the length of the smaller base equals the altitude. Find the ratio of the length of the larger base to the length of the smaller base.

Solution.

Without loss of generality, consider the figure below where $r = \frac{\overline{CD}}{\overline{AB}} > 1$ is the ratio of the length of the larger base to the length of the smaller base and $\overline{AB} = 1$ unit.



Then $\overline{OD} = \frac{1}{2}(r-1)$, and in $\triangle BOC$ we have $(r - \frac{1}{2}(r-1))^2 + 1 = r^2$, or $3r^2 - 2r - 5 = (3r-5)(r+1) = 0$. Hence $r = \frac{5}{3}$.

Remark. This problem comes from the 1953 AHSME.

3. Consider the Diophantine equation $x(2x^2 + 3x + 3) = y^3 - 1$. Prove that it does not have a solution (x, y) in positive integers, or find such a solution if it does.

Solution.

The equation is equivalent to $x^3 + (x+1)^3 = y^3$. By Fermat's Last Theorem, the equation $x^n + y^n = z^n$ has no solutions in positive integers x, y, z if the integer n is larger than 2. Hence, the given Diophantine equation has no solution.

4. The Fibonacci and Lucas numbers are sequences defined by the following initial conditions and second order recurrence relations. Let $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$ define $F_n = F_{n-1} + F_{n-2}$; let $L_0 = 2$, $L_1 = 1$, and for $n \geq 2$ define $L_n = L_{n-1} + L_{n-2}$.

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. The Binet form of the Fibonacci and Lucas numbers are given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where n is any nonnegative integer.

Let k be a fixed positive integer and define

$$U_n = \frac{F_{kn}}{L_k},$$

where n is any nonnegative integer. Find U_0 and U_1 and find a second order recurrence relation involving L_k satisfied by the sequence $\{U_n\}_{n=0}^{\infty}$.

Solution.

$$U_0 = \frac{F_0}{L_k} = 0 \quad \text{and} \quad U_1 = \frac{F_k}{L_k}.$$

We shall express U_{n+1} in terms of one or more U_n's of lower order. We have the identity

$$\begin{aligned}\alpha^{k(n+1)} - \beta^{k(n+1)} &= (\alpha^k + \beta^k)(\alpha^{kn} - \beta^{kn}) - (\alpha^{kn}\beta^k - \alpha^k\beta^{kn}) \\ &= (\alpha^k + \beta^k)(\alpha^{kn} - \beta^{kn}) - (\alpha\beta)^k(\alpha^{k(n-1)} - \beta^{k(n-1)})\end{aligned}$$

Multiplication of both sides by $\frac{1}{\sqrt{5}(\alpha^k+\beta^k)}$ and use of the definition of the U_n's, and of the fact that $\alpha\beta = -1$, gives

$$U_{n+1} = L_k U_n - (-1)^k U_{n-1}.$$

5. If your calculator is set to radian mode, and you enter any number and then repeatedly push the "cosine" button, the displayed value will converge to $0.73908\ldots$. Call this number d. If $f(x) = x - \cos x$, then $f(d) = 0$. The number d can be expressed as a series in odd powers of π:

$$d = \sum_{n=0}^{\infty} a_n \pi^{2n+1}.$$

Find a_0 and a_1.

Solution.

The function f is increasing on the interval $(-\frac{\pi}{2}, \frac{3\pi}{2})$, so has an inverse on this interval. Call the inverse g. Then $g(0) = d$ and $g(\frac{\pi}{2}) = \frac{\pi}{2}$. We will develop the Taylor series expansion of $g(x)$ around $\frac{\pi}{2}$.

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+\sin(g(x))}.$$

So, $g'(\frac{\pi}{2}) = \frac{1}{2}$.

$$g''(x) = -(1+\sin(g(x)))^{-2}\cos(g(x))\left(\frac{1}{1+\sin(g(x))}\right) = \frac{-\cos(g(x))}{(1+\sin(g(x)))^3}.$$

So, $g''(\frac{\pi}{2}) = 0$.
One more differentiation yields

$$g'''(x) = \frac{\sin(g(x))}{(1+\sin(g(x)))^4} + \cos(g(x))(\text{bounded terms}).$$

So, $g'''(\frac{\pi}{2}) = \frac{1}{16}$, and we have

$$g(x) = \frac{\pi}{2} + \frac{1}{2}\left(x - \frac{\pi}{2}\right) + \frac{1}{16}\frac{(x-\frac{\pi}{2})^3}{3!} + \cdots,$$

making

$$d = g(0) = \frac{\pi}{2} + \frac{1}{2}\left(0 - \frac{\pi}{2}\right) + \frac{1}{16}\frac{(0-\frac{\pi}{2})^3}{3!} + \cdots = \frac{\pi}{4} - \frac{\pi^3}{768} + \cdots,$$

so $a_0 = \frac{1}{4}$ and $a_1 = -\frac{1}{768}$.

Remark. See "The Dottie Number," *Math. Mag.*, 80 (2007), 73–74.

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Session II

1. Let S be the collection of ordered pairs (x, y) in $[0,1]\times[0,1]$ such that either x or y is irrational. Prove or disprove that for any two distinct ordered pairs in S, we can find a path in S connecting the two points.

Solution.

We can always find a path in S between any two distinct ordered pairs in S. To prove this, let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two distinct points in S. We need to consider 4 cases.

Case 1. If x_1 and y_2 are irrational, then draw the vertical line $x = x_1$ and the horizontal line $y = y_2$. These lines intersect at the point $P_3 = (x_1, y_2)$. Start at P_1 and follow the vertical line to P_3 and then follow the horizontal line to P_2. This is a path in S that connects P_1 and P_2.

Case 2. If x_2 and y_1 are irrational, then draw the horizontal line $y = y_1$ and the vertical line $x = x_2$. These lines intersect at the point $P_3 = (x_2, y_1)$. Start at P_1 and follow the horizontal line to P_3 and then follow the vertical line to P_2. This is a path in S that connects P_1 and P_2.

Case 3. If x_1 and x_2 are irrational, then draw the two vertical lines $x = x_1$ and and $x = x_2$. Also, draw the horizontal line $y = \sqrt{2}/2$. Let $P_3 = (x_1, \sqrt{2}/2)$ and $P_4 = (x_2, \sqrt{2}/2)$. Start at P_1 and go to P_3 along the vertical line $x = x_1$. Then go from P_3 to P_4 along the horizontal line. Finally, go from P_4 to P_2 along the vertical line $x = x_2$. This is a path in S that connects P_1 and P_2.

Case 4. If y_1 and y_2 are irrational, then draw the two horizontal lines $y = y_1$ and and $y = y_2$. Also, draw the vertical line $x = \sqrt{2}/2$. Let $P_3 = (\sqrt{2}/2, y_1)$ and $P_4 = (\sqrt{2}/2, y_2)$. Start at P_1 and go to P_3 along the horizontal line $y = y_1$. Then go from P_3 to P_4 along the vertical line. Finally, go from P_4 to P_2 along the horizontal line $y = y_2$. This is a path in S that connects P_1 and P_2.

This completes the proof.

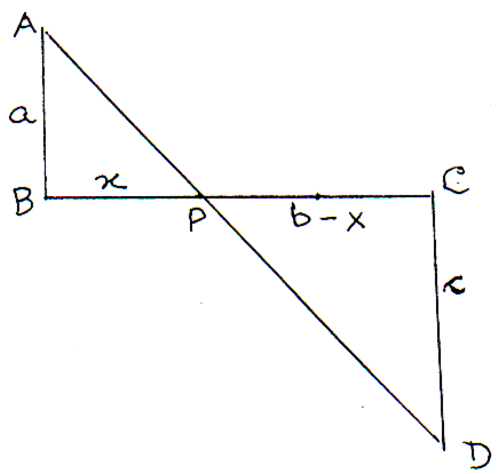
2. If a, b, c are positive real numbers, find the value of x that minimizes the function

$$f(x) = \sqrt{a^2 + x^2} + \sqrt{(b-x)^2 + c^2}.$$

(Hint: Think geometrically.)

Solution.

The simplest solution is to use geometry. Consider the figure below where $AB = a$, $BC = b$, $CD = c$, and $BP = x$.



We note that $f(x) = AP + PD$, which is a minimum when P lies at the intersection of lines BC and AD. Then

$$\frac{BP}{PC} = \frac{x}{b-x} = \frac{a}{c}.$$

Hence,

$$x = \frac{ab}{a+c}.$$

3. A sequence of 2×2 matrices, $\{M_n\}_{n=1}^{\infty}$, is defined as follows:

$$M_n = \begin{pmatrix} m_{11} = \frac{1}{(2n+1)!} & m_{12} = \frac{1}{(2n+2)!} \\ m_{21} = \sum_{k=0}^{n} \frac{(2n+2)!}{(2k+2)!} & m_{22} = \sum_{k=0}^{n} \frac{(2n+1)!}{(2k+1)!} \end{pmatrix}.$$

For each n, let $\det M_n$ denote the determinant of M_n. Determine the value of

$$\lim_{n\to\infty} \det M_n.$$

Solution.

$$\det M_n = m_{11}m_{22} - m_{12}m_{21} = \sum_{k=0}^{n} \frac{1}{(2k+1)!} - \sum_{k=0}^{n} \frac{1}{(2k+2)!}$$

$$= \sum_{k=1}^{2n+2} (-1)^{k+1} \frac{1}{k!} = \sum_{k=0}^{2n+2} (-1)^{k+1} \frac{1}{k!} - (-1) = 1 - \sum_{k=0}^{2n+2} (-1)^{k} \frac{1}{k!}.$$

Hence,

$$\lim_{n\to\infty} \det M_n = 1 - \lim_{n\to\infty} \sum_{k=0}^{2n+2} (-1)^{k} \frac{1}{k!} = 1 - e^{-1}.$$

4. Evaluate the integral

$$I = \int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2} dx.$$

Solution.

$$I = \int_{\frac{1}{2}}^{1} \frac{\ln x}{1+x^2} dx + \int_{1}^{2} \frac{\ln x}{1+x^2} dx = \int_{\frac{1}{2}}^{1} \frac{\ln x}{1+x^2} dx + \int_{1}^{\frac{1}{2}} \frac{-\ln u}{1+u^{-2}} \left(\frac{-1}{u^2} du \right)$$

$$= \int_{\frac{1}{2}}^{1} \frac{\ln x}{1+x^2} dx - \int_{\frac{1}{2}}^{1} \frac{\ln u}{1+u^2} du = 0.$$

5. A function f has the following properties:

(a) For all $x \geq 1$, $f(x)$ is a positive, differentiable, decreasing function;
(b) Whenever x equals a natural number k, we set $f(k) = f_k$, an element of a numerical sequence;
(c) The series $\sum_{k=1}^{\infty} f_k$ diverges to ∞;
(d) F is an arbitrary antiderivative of f, but with a fixed constant of integration, and is defined for all $x \geq 1$;
(e) $\lim_{x\to\infty} F(x) = \infty$.

Prove that

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} f_k - F(n) \right)$$

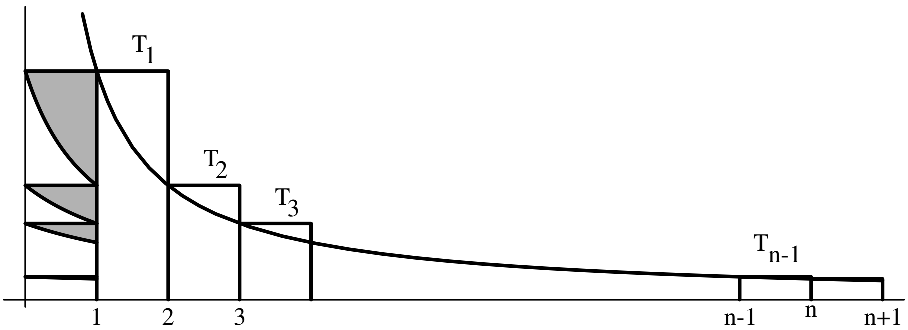
is finite.

Solution.

By the Mean Value Theorem, for each natural number k there is a constant M_k such that $k < M_k < k+1$ and

$$F(k+1) - F(k) = ((k+1) - k)F'(M_k) = f(M_k).$$

The MVT applies here because f differentiable on $[1, \infty)$ implies that f is integrable on $[1, x]$ for all real $x > 1$, which implies that F is continuous on $[1, x]$ by the Fundamental Theorem of Calculus.



Since f decreases on $[k, k+1]$, then $f_k \geq F(k+1) - F(k) \geq f_{k+1}$. Then

$$\sum_{k=1}^{n+1} f_k - F(n+1) - \left(\sum_{k=1}^{n} f_k - F(n) \right) = f_{n+1} - (F(n+1) - F(n)) \leq 0$$

from the preceding line. Since n is arbitrary, this shows that the sequence $\{c_n\}_{n=1}^{\infty}$, where

$$c_n = \sum_{k=1}^{n} f_k - F(n)$$

is a decreasing sequence.

Finally, since f is Riemann integrable over any interval $[1, n]$ and since f is positive on this interval, then from the proof of the Integral Test, one has (by definition of c_n)

$$\int_1^n f(x)dx = \sum_{k=1}^{n-1} f_k = c_n + F(n) - f_n,$$

or

$$F(n) - F(1) \leq c_n + F(n) - f_n.$$

Hence,

$$c_n \geq f_n - F(1) > -F(1),$$

and the terms of $\{c_n\}_{n=1}^{\infty}$ are bounded below by $-F(1)$. It follows from the Bounded Monotone Sequence Theorem that $\lim_{n\to\infty} c_n$ is finite.