2005 Missouri Collegiate Mathematics Competition

Session I

1. Find the point in the first quadrant on the graph of $y = 7 - x^2$ such that the distance between the x- and y- intercepts of the tangent line at the point is minimum.

Solution.

Let $(a, 7 - a^2)$, $a > 0$, be a point in the first quadrant on the graph of the function $y = 7 - x^2$. The slope of the tangent line at this point is $-2a$, and the equation of the tangent line is $y - (7 - a^2) = -2a(x - a)$. The x-intercept of this line is $((a^2 + 7)/(2a), 0)$, and the y-intercept is $(0, a^2 + 7)$. The distance squared between these two points is

$$s^2 = \frac{(a^2+7)^2}{4a^2} + (a^2+7)^2 = \frac{(a^2+7)^2(4a^2+1)}{4a^2}.$$

The derivative of s^2 with respect to a is

$$\frac{(a^2+7)(8a^4+a^2-7)}{2a^3},$$

and this factors as

$$\frac{(a^2+7)(8a^2-7)(a^2+1)}{2a^3}.$$

The critical point in the first quadrant occurs at

$$a = \sqrt{\frac{7}{8}} = \frac{\sqrt{14}}{4}.$$

Checking the first or second derivative will show that this critical value produces a minimum value for s^2, so the point that makes the distance between the x- and y- intercepts of the tangent line minimum is

$$\left(\frac{\sqrt{14}}{4}, \frac{49}{8}\right).$$

The function $y = 9 - x^2$ would also work, and then the minimum occurs at $(1, 8)$.

2. Let $M(n, k)$ be the number of mappings from a set X of n distinct objects onto a set Y of k distinct objects, and let $P(n, k)$ denote the number of partitions of a set of n distinct objects into k nonempty subsets. Determine the relationship between $M(n, k)$ and $P(n, k)$ and use it to show that $M(n, k)$ is a multiple of 24 whenever $k > 3$.

Solution.

Consider any particular partition of the elements of X into k nonempty subsets. Number these subsets from 1 to k. Similarly, number the elements in Y from 1 to k, but keep this numbering fixed. Any numbering of the sets of X then defines one surjection from X to Y. There are $k!$ possible numberings of X, so this leads to $k!$ surjections for a given partition. But there are $P(n,k)$ possible partitions, so we have

$$M(n,k) = k!P(n,k).$$

For any $k > 3$ one has that $4!$ divides $k!$, and thus 24 divides $M(n,k)$.

3. Suppose that f is a polynomial of positive degree n with integer coefficients. Prove that there are infinitely many integers x for which $f(x)$ is composite. (Here, composite means those integers, positive or negative, whose absolute value is not 1 or a prime; thus, -4 and 6 are composite, while 1 and -2 are not.)

Solution I.

Let $b = f(0)$ be the constant term of the polynomial. If $b = 0$, then $f(x)$ is even for all even integers x. The only even integers which are not composite are -2, 0, and 2. But, since f has degree n, it can only take on any of those values at most n times, and therefore, there must be infinitely many even x for which $f(x)$ is even and composite. Similarly, if b is divisible by any prime p, then $f(x)$ will be divisible by p for any x which is a multiple of p. And since f can only take on the values $-p$, 0, or p at most n times each, there will be infinitely many x for which $f(x)$ is a composite multiple of p. This leaves the cases of $b = \pm 1$. Note that, for any fixed integer t, if we think of $f(x+t)$ as a polynomial in x, then as x runs through the integers, it will take on all the same values as $f(x)$ does. So $f(x+t)$ satisfies the desired property if and only if $f(x)$ does. To complete the proof, we need only show that there exists some t for which $f(x+t)$ has a constant term other than ± 1 (allowing us to use one of the above arguments to conclude that $f(x+t)$, and hence $f(x)$, has the desired property). However, the constant term in the polynomial $f(x+t)$ (again thought of as a polynomial in x) is $f(t)$. And since f has positive degree n, $f(t)$ can only take on the values 1 or -1 at most n times each, so there must be a positive integer t for which $f(t)$ is not equal to ± 1.

Solution II.

Suppose, to the contrary, that there are only finitely many x $(x_1, x_2, \ldots, x_M)$, such that $f(x)$ is composite. Then for <u>all</u> $x > x_M$, $f(x)$ is prime:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N = p, \quad c_k \in \mathbb{N} \text{ for all } k.$$

For x sufficiently large, the polynomial is dominated by the leading term, so in this region the polynomial becomes monotonic. Let X be in this region and exceed x_M, and set $F(X) = P$ (a prime). Also, $(P+1)X$ is in this region; set $f((P+1)X) = Q$. By earlier remarks $|Q| > |P|$. Then

$$|f((P+1)X) - f(X)| = \left| \sum_{k=1}^{N} c_k((P+1)^k - 1)X^k \right| = |Q - P|.$$

But P divides each term in the summation, so P must divide Q also, a contradiction, since Q is prime. Hence, the initial supposition is false.

4. Determine the value of the integral

$$I(\theta) = \int_{-1}^{1} \frac{\sin\theta \, dx}{1 - 2x\cos\theta + x^2},$$

and locate those points $0 \le \theta \le 2\pi$, where $I(\theta)$ is discontinuous.

Solution.

Replace $1 = \cos^2\theta + \sin^2\theta$ and let $u = x - \cos\theta$, $du = dx$. Then the integral becomes

$$I(\theta) = \int_{-1-\cos\theta}^{1-\cos\theta} \frac{\sin\theta\, du}{u^2 + \sin^2\theta} = \mathrm{Tan}^{-1}\left(\frac{u}{\sin\theta}\right)\Bigg|_{-1-\cos\theta}^{1-\cos\theta}$$

$$= \mathrm{Tan}^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) - \mathrm{Tan}^{-1}\left(\frac{-1-\cos\theta}{\sin\theta}\right). \qquad (\sin\theta \neq 0)$$

$$= \mathrm{Tan}^{-1}\left(\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right) - \mathrm{Tan}^{-1}\left(\frac{-2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$

$$= \mathrm{Tan}^{-1}(\tan(\theta/2)) - \mathrm{Tan}^{-1}(-\cot(\theta/2))$$

$$= \mathrm{Tan}^{-1}(\tan(\theta/2)) - \mathrm{Tan}^{-1}(\tan(\pi/2 + \theta/2)).$$

Here, we recall that the range of $\mathrm{Tan}^{-1}\theta$ is $(-\pi/2, \pi/2)$. As an example, $\mathrm{Tan}^{-1}(\tan(20°)) = 20°$, $\mathrm{Tan}^{-1}(\tan(100°)) = -80°$, $\mathrm{Tan}^{-1}(\tan(220°)) = 40°$, and $\mathrm{Tan}^{-1}(\tan(290°)) = -70°$.

Thus, if $0 < \theta < \pi$, then $I(\theta) = \theta/2 - (-(\pi/2 - \theta/2)) = \pi/2$.

But, if $\pi < \theta < 2\pi$, then $I(\theta) = -(\pi - \theta/2) - (\theta/2 - \pi/2) = -\pi/2$.

Therefore, the function $I(\theta)$ is

$$I(\theta) = \begin{cases} 0, & \text{for } \theta = 0 \\ \frac{\pi}{2}, & \text{for } 0 < \theta < \pi \\ 0, & \text{for } \theta = \pi \\ -\frac{\pi}{2}, & \text{for } \pi < \theta < 2\pi \\ 0, & \text{for } \theta = 2\pi. \end{cases}$$

The points of discontinuity occur at 0, π, and 2π.

5. Prove that in the MacLaurin series for $\tan\theta$, $-\pi/2 < \theta < \pi/2$, every coefficient is non-negative.

Solution.

Let D_n denote the nth derivative of $\tan\theta$. We shall prove that

$$\begin{cases} D_{2k} = \tan\theta P_k(\tan^2\theta) & (1a) \\ D_{2k+1} = \sec^2\theta Q_k(\tan^2\theta), & (1b) \end{cases}$$

where $P_k(\tan^2\theta)$, $Q_k(\tan^2\theta)$ are polynomials of degree k in $\tan^2\theta$ with no terms missing and all coefficients positive. Equations (1) are true for $k=1$; assume they are true for $k=K$. Then

$$D_{2K+2} = \frac{d}{d\theta}D_{2K+1} = \sec^4\theta Q'_K(\tan^2\theta) + 2\sec^2\theta\tan\theta Q_K(\tan^2\theta).$$

Polynomial $Q'_K(\tan^2\theta)$ is of degree $2K-1$ in $\tan\theta$, all terms are odd powers, no odd powers are missing, and all coefficients are positive (by hypothesis). Write $Q'_K(\tan^2\theta) = \tan\theta f_{K-1}(\tan^2\theta)$, and therefore

$$D_{2K+2} = \tan\theta\Big(\sec^4\theta f_{K-1}(\tan^2\theta) + 2\sec^2\theta Q_K(\tan^2\theta)\Big)$$

$$= \tan\theta\Big((1+\tan^2\theta)^2 f_{K-1}(\tan^2\theta) + 2(1+\tan^2\theta)Q_K(\tan^2\theta)\Big)$$

$$= \tan\theta P_{K+1}(\tan^2\theta). \qquad (2)$$

Polynomial $P_{K+1}(\tan^2\theta)$ has properties analogous to those of $P_K(\tan^2\theta)$ because only additions and multiplications were involved in constructing it.

Similarly, we obtain

$$D_{2K+3} = \frac{d}{d\theta}D_{2K+2} = \tan\theta\sec^2\theta P'_{K+1}(\tan^2\theta) + \sec^2\theta P_{K+1}(\tan^2\theta)$$

$$= \sec^2\theta\Big(\tan\theta P'_{K+1}(\tan^2\theta) + P_{K+1}(\tan^2\theta)\Big)$$

$$= \sec^2\theta Q_{K+1}(\tan^2\theta). \qquad (3)$$

Therefore, by the Principle of Mathematical Induction, equations (1) are true for all positive integers.

Equation (1a) gives $D_{2k} = 0$ for $\theta = 0$ and all k, while equation (1b) gives $D_{2k+1} > 0$ for $\theta = 0$ and all k. Since the MacLaurin series for $\tan\theta$ is

$$\tan\theta = \sum_{k=0}^{\infty}\frac{1}{k!}D_k(0)\theta^k, \qquad (-\pi/2 < \theta < \pi/2)$$

then this reduces to

$$\tan\theta = \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}D_{2k+1}(0)\theta^{2k+1},$$

where $c_k > 0$ for all k. The first few terms are computed to be

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \frac{62\theta^9}{2835} + \cdots .$$

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Session II

1. If $a < b < c$, $f'(x)$ is strictly increasing on (a, c), and $f(x)$ is continuous on $[a, c]$, then show that

$$(b-a)f(c) + (c-b)f(a) > (c-a)f(b).$$

Solution.

Under the hypotheses, the Mean Value Theorem for derivatives is valid, so there are numbers α and β such that

$$\frac{f(c)-f(b)}{c-b} = f'(\beta), \qquad b < \beta < c$$

and

$$\frac{f(b)-f(a)}{b-a} = f'(\alpha), \qquad a < \alpha < b.$$

But f' is strictly increasing, so $f'(\beta) > f'(\alpha)$ and, hence,

$$\frac{f(c)-f(b)}{c-b} > \frac{f(b)-f(a)}{b-a}.$$

An elementary rearrangement gives the desired result.

2. Find all integer solutions (x, y) to the equation $xy = 5x + 11y$.

Solution.

First, if we re-write the quation $xy = 5x + 11y$ as $xy - 5x - 11y = 0$, we can add 55 to both sides to get $xy - 5x - 11y + 55 = 55$. So we have

$$(x-11)(y-5) = 55.$$

Now, x and y are supposed to be integers, so $x - 11$ and $y - 5$ should also be integers. Thus, the question is, how many different pairs of integers can we find whose product is 55? There are eight pairs:

$$(55, 1),\ (-55, -1),\ (1, 55),\ (-1, -55),\ (5, 11),\ (-5, -11),\ (11, 5),\ (-11, -5).$$

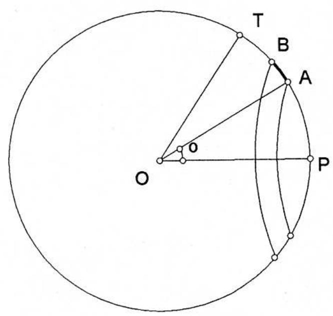
If $x - 5$ is to represent the first number and $y - 11$ is to represent the second number, then solving for x and y in each pair yields the eight possible solutions

$$\begin{array}{llll} x = 66,\ y = 6 & x = -44,\ y = 4 & x = 12,\ y = 60 & x = 10,\ y = -50 \\ x = 16,\ y = 16 & x = 6,\ y = -6 & x = 22,\ y = 10 & x = 0,\ y = 0. \end{array}$$

From Robert Roe, University of Missouri-Rolla.

3. Define a circle of radius r and center P on a sphere to be the locus of points on the surface of the sphere that are a distance r from P, where distance is the usual Euclidean distance in $\mathbb{R}^3$. When r is less than the diameter of the sphere, this circle divides the sphere into two spherical segments, or "caps". Show that the area of the cap containing the center point P is πr^2.

Solution.



Let O and R denote the center and the radius of the sphere. Any section of the (hollow) sphere by a plane perpendicular to OP is a circle say Γ. Let T be a point on the circle Γ, for which $PT = r$. When arc PT is revolved about the axis OP, it will generate the spherical cap whose area is to be computed. Let $A(R, \theta)$ and $B(R, \theta + \delta\theta)$ be nearby points on the arc PT. The length of the arc AB is $R \cdot \delta\theta$. When we rotate arc AB about the line OP, the area of the surface generated is

$$2\pi(R \sin\theta)(R \cdot \delta\theta).$$

If $\angle POT = \alpha$, then $r = 2R \sin\alpha/2$. The area of the cap is

$$\int_0^\alpha 2\pi R \sin\theta R \, d\theta = 2\pi R^2 \int_0^\alpha \sin\theta R \, d\theta = 2\pi R^2(1 - \cos\alpha) = 4\pi R^2 \sin^2\left(\frac{\alpha}{2}\right) = \pi r^2.$$

4. Let $p > 2$ be a prime. Prove or disprove that all prime divisors of $2^p - 1$ have the form $2kp + 1$.

Solution.

The fact that all prime divisors of $2^p - 1$, where $p > 2$ is prime, have the form $2kp + 1$ was known to Fermat. Let q be a prime divisor of $2^p - 1$. Then $2^p \equiv 1 \pmod{q}$. Since p is prime and $2^1 \not\equiv 1 \pmod{q}$, it follows that the order of 2 mod q is p. By Fermat's little theorem, i.e., $2^{q-1} \equiv 1 \pmod{q}$, we get $p \mid q - 1$. Thus, there exists j such that $jp = q - 1$. Since p is odd and $q - 1$ is even, $j = 2k$ for some integer k.

5. Suppose that $f\colon[0,\infty)\to[0,\infty)$ is a differentiable function with the property that the area under the curve $y=f(x)$ from $x=a$ to $x=b$ is equal to the arclength of the curve $y=f(x)$ from $x=a$ to $x=b$. Given that $f(0)=5/4$, and that $f(x)$ has a minimum value on the interval $(0,\infty)$, find that minimum value.

Solution.

The area under the curve $y=f(x)$ from $x=a$ to $x=b$ is

$$\int_a^b f(t)\,dt,$$

and the arclength of the curve $y=f(x)$ from $x=a$ to $x=b$ is

$$\int_a^b \sqrt{1+(f'(t))^2}\,dt.$$

Therefore,

$$\int_a^b f(t)\,dt=\int_a^b \sqrt{1+(f'(t))^2}\,dt$$

for all nonnegative a and b. In particular, we can write

$$\int_0^x f(t)\,dt=\int_0^x \sqrt{1+(f'(t))^2}\,dt$$

for all nonnegative x. Both sides of the above equation define a function of x, and since they are equal, their derivatives are equal; their derivatives are given by the Second Fundamental Theorem of Calculus:

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right)=\frac{d}{dx}\left(\int_0^x \sqrt{1+(f'(t))^2}\,dt\right),$$

i.e.,

$$f(x)=\sqrt{1+(f'(x))^2}.$$

So, we are looking for a function y which satisifies the differential equation

$$y=\sqrt{1+(y')^2}.$$

This equation is separable:

$$\begin{aligned} y=\sqrt{1+(y')^2} &\Rightarrow y^2=1+(y')^2\\ &\Rightarrow (y')^2=y^2-1\\ &\Rightarrow y'=\sqrt{y^2-1}\\ &\Rightarrow \frac{dy}{\sqrt{y^2-1}}=dx. \end{aligned}$$

Integrating both sides yields

$$\int \frac{dy}{\sqrt{y^2-1}} = \int dx \Rightarrow \ln\left|y+\sqrt{y^2-1}\right| = x + C$$

(where the first integral is evaluated using the trig substitution $y = \sec\theta$ and the two arbitrary constants of integration are combined into one constant on the right hand side). Next, since $f(0) = 5/4$ is positive, we can drop the absolute value, and solve for y:

$$\begin{aligned}
\ln\bigl(y+\sqrt{y^2-1}\bigr) = x + C \Rightarrow y+\sqrt{y^2-1} = e^{x+C} &= Ae^x \quad (\text{where } A = e^C) \\
&\Rightarrow \sqrt{y^2-1} = Ae^x - y \\
&\Rightarrow y^2 - 1 = (Ae^x - y)^2 = A^2e^{2x} - 2Aye^x + y^2 \\
&\Rightarrow -1 = A^2e^{2x} - 2Aye^x \\
&\Rightarrow 2Aye^x = A^2e^{2x} + 1 \\
&\Rightarrow y = \frac{A^2e^{2x}+1}{2Ae^x} = \frac{A}{2}e^x + \frac{1}{2A}e^{-x}.
\end{aligned}$$

Using $f(0) = 5/4$, we find

$$\frac{5}{4} = \frac{A}{2} + \frac{1}{2A} \Rightarrow A = \frac{1}{2} \text{ or } 2.$$

This gives two possible functions:

$$y = \frac{1}{4}e^x + e^{-x} \text{ or } y = e^x + \frac{1}{4}e^{-x}.$$

This latter has a minimum at $x = -\ln 2$, which is not positive, so we reject that function. The former has a minimum at $x = \ln 2$, and the y value is 1.

Note: One could also deduce from the differential equation $y' = \sqrt{y^2-1}$ that at the minimum value, since $y' = 0$, the y-value must be 1.